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SUR LA REPRESENTATION INTEGRALE DES CERTAINES OPERATIONS LINEAIRES. IV OPERATIONS LINEAIRES SUR L'ESPACE L_a^p

N. DINCULEANU ET C. FOIAS

1. Introduction. La première partie de cet article a un caractère introductif; on reprend l'étude des mesures vectorielles sur un espace localement compact, présenté dans **(8)** pour les espaces compacts.

Dans la seconde partie on donne un théorème de représentation intégrale d'une mesure vectorielle par rapport à sa variation (théorème 2), duquel on déduit une généralisation du théorème de Lebesgue–Nikodym (théorème 3) et une généralisation du théorème de Lebesgue sur la décomposition d'une mesure (théorème 5). Ces théorèmes ont été déjà démontrés par les auteurs sous des conditions un peu plus restrictives, **(10)** et **(13)**, mais ici les démonstrations sont nouvelles, et l'ordre dont on les déduit l'un de l'autre, est inversée.

Du théorème 2 on déduit aussi un théorème qui donne une condition nécessaire et suffisante pour qu'une application linéaire de l'espace $L_a^p(\nu)$ dans un espace de Banach puisse être représentée sous une forme intégrale (théorème 6). De ce théorème, qui nous semble nouveau, on déduit le théorème de C. T. Ionescu Tulcea **(15)** concernant les fonctionnelles linéaires et continues sur l'espace $L_a^p(\nu)$ et un thèoréme concernant les applications linéaires et continues de l'espace $L_a^1(\nu)$ dans un espace de Banach, donné par l'un des auteurs dans **(6)**.

I. MESURES VECTORIELLES

1. Préliminaires. Soient T un espace localement compact et $\mathfrak{B}$ le clan des parties boréliennes relativement compactes de T. Soient $\mathfrak{E} = (E(t))_{t\in T}$ une famille d'espaces de Banach et X un espace de Banach.

Désignons par $\mathfrak{C}(\mathfrak{E})$ l'ensemble des champs de vecteurs $\mathbf{x}$ définis sur T, tels que $\mathbf{x}(t) \in E(t)$ quel que soit $t \in T$. La norme d'un élément a appartenant à l'un des espaces $E(t)$ ou X sera notéé par $|a|$. Si $\mathbf{x} \in \mathfrak{C}(\mathfrak{E})$, on désigne par $|\mathbf{x}|$ la fonction numérique $t \to |\mathbf{x}(t)|$. Pour tout $\mathbf{x} \in \mathfrak{C}(\mathfrak{E})$ et tout $A \subset T$ on pose

$$||\mathbf{x}||_A = \sup_{t\in A} |\mathbf{x}(t)|$$

et $||\mathbf{x}|| = ||\mathbf{x}||_T$.

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Supposons qu'il existe une famille fondamentale[1] $\mathfrak{a} \subset \mathfrak{C}(\mathfrak{E})$ de champs de vecteurs continus.

Un champ de vecteurs $\mathbf{x} \in \mathfrak{C}(\mathfrak{E})$ est continu (part rapport à $\mathfrak{a}$) au point $t_0 \in T$, si pour tout nombre $\epsilon > 0$, il existe un voisinage V de t_0 et un champ de vecteurs $\mathbf{y} \in \mathfrak{a}$, tels que

$$|\mathbf{x}(t) - \mathbf{y}(t)| < \epsilon$$

pour $t \in V$.

Si un champ de vecteurs $\mathbf{x}$ est continu au point t_0, alors la fonction numérique $|\mathbf{x}|$ est continue en t_0.

Lorsque tous les espaces $E(t)$ sont identiques à un espace de Banach E, on prend pour $\mathfrak{a}$ l'ensemble des applications constantes de T dans E, et on identifie $\mathfrak{a}$ avec E. Dans ce cas la continuité par rapport à $\mathfrak{a}$ est la continuité habituelle.

Dans ce qui suit, (à l'exception du théorème 6), on supposera que $\mathfrak{a}$ vérifie l'axiome suivant:

(i) *Quels que soient* $A, B \in \mathfrak{B}$, $A \subset B$, $\mathbf{x} \in \mathfrak{a}$ *et* $\epsilon > 0$, *il existe* $\mathbf{y} \in \mathfrak{a}$ *tel que* $\varphi_A\mathbf{x} = \varphi_A\mathbf{y}$ *et* $||\mathbf{y}||_B < ||\mathbf{x}||_A + \epsilon$.

Dans le cas où $E(t) = E$ quel que soit $t \in T$, l'axiome (i) est vérifiée.

Pour tout $A \in \mathfrak{B}$, la fonction $\mathbf{x} \to ||\mathbf{x}||_A$ est une séminorme sur $\mathfrak{a}$ Si $A \subset B$, alors $||\mathbf{x}||_A < ||\mathbf{x}||_B$. Pour tout $A \in \mathfrak{B}$, on désigne par $\mathfrak{a}_A$ l'espace vectoriel $\mathfrak{a}$ muni de la topologie de la convergence uniforme sur A, définie par la séminorme $||\mathbf{x}||_A$, est par $\mathfrak{L}(\mathfrak{a}_A, X)$, l'espace de Banach des applications linéaires et continues de $\mathfrak{a}_A$ dans X, munit de la topologie définie par la norme

$$||U||_A = \sup_{||\mathbf{x}||_A \leq 1} |U\mathbf{x}|.$$

Si $A \subset B$, alors $\mathfrak{L}(\mathfrak{a}_A, X) \subset \mathfrak{L}(\mathfrak{a}_B, X)$ et $||U||_A \geqslant ||U||_B$.

Lorsqu'on écrit $\mathfrak{a}$, on entend que $\mathfrak{a}$ est muni de la topologie d'espace localement convexe séparé définie par la famille des séminormes $(||\mathbf{x}||_A)_{A \in \mathfrak{B}}$; $\mathfrak{L}(\mathfrak{a}, X)$ est alors l'espace des applications linéaires et continues de $\mathfrak{a}$ dans X. On a

$$\mathfrak{L}(\mathfrak{a}, X) = \bigcup_{A \in \mathfrak{B}} L(\mathfrak{a}_A, X).$$

2. Mesures vectorielles. On appelle *fonction vectorielle d'ensemble*, toute application $\mathbf{m}$ de $\mathfrak{B}$ dans $\mathfrak{L}(\mathfrak{a}, X)$, vérifiant l'axiome suivant:

(j) *pour tout* $A \in \mathfrak{B}$ *et* $\mathbf{x}, \mathbf{y} \in \mathfrak{a}$ *tels que* $\varphi_A\mathbf{x} = \varphi_A\mathbf{y}$, *on a*

$$\mathbf{m}(A)\mathbf{x} = \mathbf{m}(A)\mathbf{y}.$$

Si $E(t) = E$ quel que soit $t \in T$, la condition (j) est superflue.

[1]Une famille fondamentale de champs de vecteurs continus $\mathfrak{a}$ est un sous-espace vectoriel de $\mathfrak{C}(\mathfrak{E})$ tel que: la fonction numérique $t \to |\mathbf{x}(t)|$ soit continue quel que soit $\mathbf{x} \in \mathfrak{A}$, et l'ensemble $\{\mathbf{x}(t) | \mathbf{x} \in \mathfrak{A}\}$ soit dense dans $E(t)$ quel que soit $t \in T$. En ce qui concerne les champs de vecteurs, voir **(14)**.

PROPOSITION 1. *Si* $\mathbf{m}$ *est une fonction vectorielle d'ensemble, pour tout* $A \in \mathfrak{B}$ *on a* $\mathbf{m}(A) \in \mathfrak{L}(\mathfrak{a}_A, X)$ *et*

$$||\mathbf{m}(A)||_A = ||\mathbf{m}(A)||_B$$

quel que soit $B \in \mathfrak{B}$ *tel que* $A \subset B$.

Soit $A \in \mathfrak{B}$; on a $\mathbf{m}(A) \in \mathfrak{L}(\mathfrak{a}, X)$, donc il existe $C \in \mathfrak{B}$, tel que $||\mathbf{m}(A)||_C < +\infty$. Pour tout $B \in \mathfrak{B}$ tel que $A \subset B$ on a

$$||\mathbf{m}(A)||_B \leqslant ||\mathbf{m}(A)||_A.$$

Pour montrer l'inégalité inverse, soient $\epsilon > 0$ et $\mathbf{x} \in \mathfrak{a}$ tel que $||\mathbf{x}||_A \leqslant 1$. D'après l'axiome (i), il existe $\mathbf{y} \in \mathfrak{a}$, tel que $\varphi_A \mathbf{x} = \varphi_A \mathbf{y}$ et $||\mathbf{y}||_B \leqslant 1 + \epsilon$. D'après l'axiome (j) on a

$$|\mathbf{m}(A)\mathbf{x}| = |\mathbf{m}(A)\mathbf{y}| = (1+\epsilon)\left|\mathbf{m}(A)\frac{\mathbf{y}}{1+\epsilon}\right| \leqslant (1+\epsilon)\,||\mathbf{m}(A)||_B \leqslant +\infty$$

donc

$$||\mathbf{m}(A)||_A \leqslant (1+\epsilon)||\mathbf{m}(A)||_B$$

et comme ϵ est arbitraire, on déduit

$$||\mathbf{m}(A)||_A \leqslant ||\mathbf{m}(A)||_B.$$

En prenant $B \in \mathfrak{B}$ tel que $A \cup C \subset B$, on déduit

$$||\mathbf{m}(A)||_B \leqslant ||\mathbf{m}(A)||_C < +\infty$$

donc aussi $||\mathbf{m}(A)||_A < +\infty$, ce qui achève la démonstration.

Remarques. On écrira dans la suite $||\mathbf{m}(A)||$ au lieu de $||\mathbf{m}(A)||_A$. Si (A_n) est une suite d'ensembles de $\mathfrak{B}$, dont la réunion A_0 appartient à $\mathfrak{B}$, les normes $||\mathbf{m}(A_n)||$ seront calculés par rapport à un ensemble quelconque $B \in \mathfrak{B}$ tel que $A_0 \subset B$. Par exemple, la condition

$$\mathbf{m}(A_0) = \sum_{n=1}^{\infty} \mathbf{m}(A_n)$$

signifie que la série est convergente dans $\mathfrak{L}(\mathfrak{a}_B, X)$ et sa somme est $\mathbf{m}(A_0)$.

Définition 1. On appelle mesure vectorielle sur T, toute fonction vectorielle d'ensemble $\mathbf{m}$, telle que, pour toute suite (A_n) d'ensembles disjoints de $\mathfrak{B}$, dont la réunion appartient à $\mathfrak{B}$, on ait

$$\mathbf{m}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbf{m}(A_n).$$

On déduit que si $\mathbf{m}$ est une mesure vectorielle sur T, on a $\mathbf{m}(\phi) = 0$, et

$$\mathbf{m}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbf{m}(A_i)$$

pour toute famille finie $(A_i)_{1 \leqslant i \leqslant n}$ d'ensembles disjoints de $\mathfrak{B}$.

Définition 2. On dit qu'une mesure vectorielle $\mathbf{m}$ sur T est régulière, si pour tout $A \in \mathfrak{B}$ et tout $\epsilon > 0$, il existe un ensemble compact $K \subset A$ et un ensemble ouvert relativement compact $G \supset A$, tels que si $A' \in \mathfrak{B}$ et $K \subset A' \subset G$, on ait

$$||\mathbf{m}(A) - \mathbf{m}(A')|| \leqslant \epsilon.$$

Remarques. D'après la définition 1, une fonction positive d'ensemble μ est une mesure positive, si μ est dénombrablement additive et $\mu(A) < +\infty$ pour tout $A \in \mathfrak{B}$. Une mesure complexe régulière μ sera identifiée dans la suite avec la mesure de Radon $\int f d\mu$ qu'elle engendre.[2]

3. Variation d'une mesure vectorielle. Soit $\mathbf{m}$ une mesure vectorielle sur T. Pour tout $A \in \mathfrak{B}$ posons

$$\mu(A) = \sup \sum_i ||\mathbf{m}(A_i)||$$

le sup. étant considéré pour toutes les familles finies (A_i) d'ensembles disjoints de $\mathfrak{B}$, dont la réunion est A.

La fonction d'ensemble μ sera appelée la *variation* de $\mathbf{m}$; elle a les propriétés suivantes:

1°. $0 \leqslant \mu(A) \leqslant +\infty$ pour $A \in \mathfrak{B}$;

2°. $\mu(A) = 0$ si et seulement si $\mathbf{m}(B) = 0$ pour tout $B \subset A$, $(A, B \in \mathfrak{B})$;

3°. $\mu(A) \leqslant \mu(B)$ si $A \subset B$, $(A, B \in \mathfrak{B})$;

4°. $$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

pour toute suite (A_n) d'ensembles disjoints de $\mathfrak{B}$, dont la réunion appartient à $\mathfrak{B}$;

5°. $||\mathbf{m}(A)|| \leqslant \mu(A)$ pour $A \in \mathfrak{B}$.

La variation de $\mathbf{m}$ est *la plus petite* fonction positive d'ensemble, dénombrablement additive, vérifiant l'inégalité 5° pour tout $A \in \mathfrak{B}$.

On dit que $\mathbf{m}$ est à *variation finie* μ, si $\mu(A) < +\infty$ quel soit que $A \in \mathfrak{B}$.

PROPOSITION 2. *Une mesure vectorielle à variation finie est régulière si et seulement si sa variation est régulière.*

La démonstration est la même que pour le cas où T est compact **(8)**.

Remarque. Si $\mathbf{m}$ est une mesure régulière complexe, alors la variation μ de $\mathbf{m}$ est égale au module $|\mathbf{m}|$ de $\mathbf{m}$ (voir **(2)**).

4. Sémivariation d'une mesure vectorielle. Soit $\mathbf{m}$ une mesure vectorielle sur T. Pour tout $A \in \mathfrak{B}$, posons

$$\bar{\mu}(A) = \sup \left| \sum_i \mathbf{m}(A_i)\mathbf{x}_i \right|$$

[2]Pour ce qui concerne les mesures de Radon, voir **(2)**.

le sup. étant considéré pour toutes les familles finies $(A_i)_{i\in I}$ d'ensembles disjoints de $\mathfrak{B}$ dont la réunion est A, et toutes les familles finies $(\mathbf{x}_i)_{i\in I}$ de champs de vecteurs appartenant à $\mathfrak{a}$, tels que $||\mathbf{x}_i||_A \leqslant 1$. En vertu des axiomes (i) et (j), la condition $||\mathbf{x}_i||_A \leqslant 1$ peut être remplacée par la condition $||\mathbf{x}_i||_B \leqslant 1$, où B est un ensemble quelconque de $\mathfrak{B}$ contenant A.

La fonction d'ensemble $\bar{\mu}$, s'appelle la *sémivariation* de $\mathbf{m}$; elle a les propriétés suivantes:

1°. $0 \leqslant \bar{\mu}(A) \leqslant +\infty$ pour $A \in \mathfrak{B}$;

2°. $\bar{\mu}(A) = 0$ si et seulement si $\mu(A) = 0$;

3°. $\bar{\mu}(A) \leqslant \bar{\mu}(B)$ si $A \subset B$, $(A, B \in \mathfrak{B})$;

4°. $\bar{\mu}$ est dénombrablement sous-additive;

5°. $||\mathbf{m}(A)|| \leqslant \bar{\mu}(A) \leqslant \mu(A)$ pour $A \in \mathfrak{B}$.

PROPOSITION 3. *Si $X = C$, on a $\bar{\mu}(A) = \mu(A)$ pour tout $A \in \mathfrak{B}$.*

Soit $A \in \mathfrak{B}$. Il reste à montrer seulement que $\mu(A) \leqslant \bar{\mu}(A)$. Pour ce faire, soient $\epsilon > 0$ et $(A_i)_{1\leqslant i\leqslant n}$ une famille finie d'ensembles disjoints de $\mathfrak{B}$, dont la réunion est A. Pour chaque i, il existe $\mathbf{x}_i \in \mathfrak{a}$ tel que $||\mathbf{x}_i||_A \leqslant 1$ et

$$||\mathbf{m}(A_i)|| \leqslant \mathbf{m}(A_i)\mathbf{x}_i + \frac{\epsilon}{n}.$$

Alors

$$\sum_{i=1}^{n} ||\mathbf{m}(A_i)|| \leqslant \sum_{i=1}^{n} \mathbf{m}(A_i)\mathbf{x}_i + \epsilon \leqslant \bar{\mu}(A) + \epsilon$$

donc

$$\mu(A) \leqslant \bar{\mu}(A) + \epsilon.$$

Comme ϵ est arbitraire, on déduit $\mu(A) \leqslant \bar{\mu}(A)$ et ceci achève la démonstration.

5. Champs de vecteurs étagés. On dit qu'un champ de vecteurs $\mathbf{x} \in \mathfrak{C}(\mathfrak{E})$ est $(\mathfrak{a}, \mathfrak{B})$-étagé, où, simplement, étagé, s'il est de la forme

$$\mathbf{x} = \sum_i \varphi_{A_i}\mathbf{x}_i \qquad (A_i \in \mathfrak{B}, \mathbf{x}_i \in \mathfrak{a}, 1 \leqslant i \leqslant n).$$

Désignons par $\mathfrak{E}_{\mathfrak{a}}(T)$, l'espace vectoriel des champs de vecteurs étagés, et par $\mathfrak{E}(T)$ l'espace vectoriel des fonctions numériques étagées.

On peut toujours écrire un champ de vecteurs étagé

$$\mathbf{x} = \sum_i \varphi_{A_i}\mathbf{x}_i,$$

de telle manière que les ensembles A_i soient disjoints. Alors

$$|\mathbf{x}| = \sum_i \varphi_{A_i}|\mathbf{x}_i|$$

est une fonction numérique étagée.

Soit $\mathbf{m}$ une mesure vectorielle à variation finie μ.

Pour tout champ de vecteurs étagé

$$\mathbf{x} = \sum_i \varphi_{A_i}\mathbf{x}_i,$$

on définit l'intégrale

$$\int \mathbf{x}\, d\mathbf{m}$$

de $\mathbf{x}$ par rapport à $\mathbf{m}$, par l'égalité

$$\int \mathbf{x} d\mathbf{m} = \sum_i \mathbf{m}(A_i)\mathbf{x}_i.$$

L'application

$$\mathbf{x} \to \int \mathbf{x}\, d\mathbf{m}$$

de $\mathfrak{E}_s(T)$ dans X est linéaire.

PROPOSITION 4. *Pour tout champ de vecteurs* $\mathbf{x} \in \mathfrak{E}_s(T)$ *on a*

$$\left|\int \mathbf{x}\, d\mathbf{m}\right| \leqslant \int |\mathbf{x}|\, d\mu.$$

Pour la démonstration voir **(8)**.

Désignons par $\mathfrak{B}'$ l'ensemble des parties $B \subset T$, telles que $A \cap B \in \mathfrak{B}$, quel que soit $A \in \mathfrak{B}$; $\mathfrak{B}'$ est une tribu.

Si $\mathbf{x} \in \mathfrak{E}_s(T)$ et $A \in \mathfrak{B}'$, alors $\varphi_A \mathbf{x} \in \mathfrak{E}_s(T)$.

PROPOSITION 5. *Soit* $\mathbf{x} \in \mathfrak{E}_s(T)$. *Pour toute suite* (A_n) *d'ensembles disjoints de* $\mathfrak{B}'$, *on a*

$$\int_{\bigcup_{n=1}^{\infty} A_n} \mathbf{x}\, d\mathbf{m} = \sum_{n=1}^{\infty} \int_{A_n} \mathbf{x}\, d\mathbf{m}.$$

6. Intégration des champs de vecteurs. Soit $\mathbf{m}$ une mesure vectorielle à variation finie μ. Pour tout $\mathbf{x} \in \mathfrak{E}_s(T)$ posons

$$N_1(\mathbf{x}) = \int |\mathbf{x}|\, d\mu.$$

L'application $\mathbf{x} \to N_1(\mathbf{x})$ est une semi-norme sur $\mathfrak{E}_s(T)$; la topologie définie par cette semi-norme est la topologie de la convergence en moyenne.

Définition 3. On dit qu'un champ de vecteurs $\mathbf{x} \in \mathfrak{C}(\mathfrak{E})$ est μ-intégrable, s'il existe une suite $(\mathbf{x}_n)$ de champs de vecteurs étagés, convergente vers $\mathbf{x}$ μ-presque partout et qui soit une suite de Cauchy pour la topologie de la convergence en moyenne.

Désignons par $L_s^1(\mu)$ l'espace vectoriel des champs de vecteurs μ-intégrables.

On déduit aussitôt que si $\mathbf{x}$ est un champ de vecteurs μ-intégrable, alors $|\mathbf{x}|$ est une fonction numérique μ-intégrable.

On considère sur $L_{\mathfrak{a}}^1(\mu)$ la topologie de la convergence en moyenne, définie par la sémi-norme

$$N_1(\mathbf{x}) = \int |\mathbf{x}| d\mu \qquad \text{pour} \quad \mathbf{x} \in L_{\mathfrak{a}}^1(\mu).$$

On montre que $L_{\mathfrak{a}}^1(\mu)$ est un espace complet pour cette topologie, et l'espace $\mathfrak{E}_{\mathfrak{a}}(T)$ est dense dans $L_{\mathfrak{a}}^1(\mu)$.

PROPOSITION 6. *Si* $(\mathbf{x}_n)$ *est une suite de chamsp de vecteurs de* $L_{\mathfrak{a}}^1(\mu)$, *dont les supports sont contenus dans un même ensemble compact, et si la suite* $(\mathbf{x}_n)$ *est uniformément convergente vers un champs de vecteurs* $\mathbf{x}$, *alors* $\mathbf{x}$ *est* μ*-intégrable et*

$$\lim_{n \to \infty} N_1(\mathbf{x}_n - \mathbf{x}) = 0.$$

La démonstration est immédiate.

Définition 4. Un champ de vecteurs $\mathbf{x} \in \mathfrak{C}(\mathfrak{E})$ est $\mathbf{m}$-intégrable si $\mathbf{x}$ est μ-intégrable.

De la proposition 4 on déduit que l'application $\mathbf{x} \to \int \mathbf{x} d\mathbf{m}$ de $\mathfrak{E}_{\mathfrak{a}}(T)$ dans X est continue pour la topologie de la convergence en moyenne. Comme $\mathfrak{E}_{\mathfrak{a}}(T)$ est dense dans $L_{\mathfrak{a}}^1(\mu)$ pour cette topologie et X est complet, l'application $\mathbf{x} \to \int \mathbf{x} d\mathbf{m}$ se prolonge uniquement, par continuité, à une application linéaire et continue de $L_{\mathfrak{a}}^1(\mu)$ dans X, notée toujours $\int \mathbf{x} d\mathbf{m}$. On a encore

$$\left| \int \mathbf{x}\, d\mathbf{m} \right| \leqslant \int |\mathbf{x}|\, d\mu \qquad \text{pour} \quad \mathbf{x} \in L_{\mathfrak{a}}^1(\mu).$$

Pour tout $\mathbf{x} \in L_{\mathfrak{a}}^1(\mu)$ on appelle $\int \mathbf{x} d\mathbf{m}$ l'intégrale de $\mathbf{x}$ par rapport à $\mathbf{m}$. On écrit aussi $L_{\mathfrak{a}}^1(\mathbf{m})$ au lieu de $L_{\mathfrak{a}}^1(\mu)$.

On désigne par $\mathfrak{B}(\mathbf{m})$ ou $\mathfrak{B}(\mu)$ le clan des parties μ-mesurables et relativement compacts de T. Si $A \in \mathfrak{B}(\mathbf{m})$ et $\mathbf{x} \in \mathfrak{a}$, alors $\varphi_A \mathbf{x} \in L_{\mathfrak{a}}^1(\mu)$. Pour tout $A \in \mathfrak{B}(\mathbf{m})$ on définit $\mathbf{m}(A)$ par l'égalité

$$\mathbf{m}(A)\mathbf{x} = \int \varphi_A \mathbf{x}\, d\mathbf{m} \qquad \text{pour} \quad \mathbf{x} \in \mathfrak{a}.$$

On déduit aussitôt

(1) $\mathbf{m}(A)\mathbf{x} = \mathbf{m}(A)\mathbf{y}$ si $\varphi_A \mathbf{x} = \varphi_A \mathbf{y}$, $A \in \mathfrak{B}(\mathbf{m})$, $\mathbf{x}, \mathbf{y} \in \mathfrak{a}$;

(2) $||\mathbf{m}(A)|| = \sup_{||\mathbf{x}||_B \leq 1} |\mathbf{m}(A)\mathbf{x}|$.

quel que soit $B \in \mathfrak{B}(\mathbf{m})$ tel que $A \subset B$;

(3) $\mathbf{m}(A)$ est dénombrablement additive sur $\mathfrak{B}(\mathbf{m})$.

On désigne par $\mathfrak{B}'(\mathbf{m})$ ou $\mathfrak{B}'(\mu)$ la tribu des parties μ-mesurables de T; $\mathfrak{B}'(\mathbf{m})$ est donc l'ensemble des parties $B \subset T$ telles que $A \cap B \in \mathfrak{B}(\mathbf{m})$ quel que soit $A \in \mathfrak{B}(\mathbf{m})$.

7. Mesures vectorielles régulières. Désignons par $\mathfrak{K}_{\mathfrak{a}}(T)$ l'espace vectoriel des champs de vecteurs $\mathbf{x} \in \mathfrak{C}(\mathfrak{E})$ continus (par rapport à $\mathfrak{a}$) et à support compact. Pour tout $A \subset T$, $\mathfrak{K}_{\mathfrak{a}}(T, A)$ est le sous-espace de $\mathfrak{K}_{\mathfrak{a}}(T)$ formé des champs de vecteurs dont les supports sont contenus dans A. Tout champ de vecteurs $\mathbf{x} \in \mathfrak{K}_{\mathfrak{a}}(T)$ peut être approché uniformément par des champs de vecteurs de $\mathfrak{E}_{\mathfrak{a}}(T)$, donc $\mathfrak{K}_{\mathfrak{a}}(T) \subset L_{\mathfrak{a}}^1(\mathbf{m})$.

PROPOSITION 7. *Si* $\mathbf{m}$ *est régulière et à variation finie, alors* $\mathfrak{K}_{\mathfrak{a}}(T)$ *est dense dans* $L_{\mathfrak{a}}^1(\mathbf{m})$.

En effet, la variation μ de $\mathbf{m}$ est une mesure positive régulière, et tout champ $\varphi_A \mathbf{x}$, $(A \in \mathfrak{B}, \mathbf{x} \in \mathfrak{a})$ peut être approché dans $L_{\mathfrak{a}}^1(\mu)$ par des champs de $\mathfrak{K}_{\mathfrak{a}}(T)$.

PROPOSITION 8. *Deux mesures vectorielles à variation finie et régulières* $\mathbf{m}$ *et* $\mathbf{m}'$ *sont égales, si et seulement si*

$$\int \mathbf{x}\, d\mathbf{m} = \int \mathbf{x}\, d\mathbf{m}' \qquad \text{pour tout } \mathbf{x} \in \mathfrak{K}_{\mathfrak{a}}(T).$$

La démonstration est la même que pour le cas où T est compact **(8)**.

On dit qu'une application linéaire U de $\mathfrak{K}_{\mathfrak{a}}(T)$ dans X est *majorée*, s'il existe une mesure de Radon positive λ sur T, telle que

$$|U(\mathbf{x})| \leqslant \int |\mathbf{x}|\, d\lambda \qquad \text{pour tout } \mathbf{x} \in \mathfrak{K}_{\mathfrak{a}}(T).$$

Il existe alors **(7)** une plus petite mesure de Radon positive ν sur T, majorante de U.

Si $\mathbf{m}$ est une mesure vectorielle régulière et à variation finie μ, et si l'on pose

$$U(\mathbf{x}) = \int \mathbf{x}\, d\mathbf{m} \qquad \text{pour } \mathbf{x} \in \mathfrak{K}_{\mathfrak{a}}(T)$$

alors U est une application linéaire de $\mathfrak{K}_{\mathfrak{a}}(T)$ dans X, majorée par μ; en outre, μ est la plus petitite mesure positive régulière (qu'on a convenu à identifier à une mesure de Radon), majorante de U.

Réciproquement, on a le théorème suivant:

THÉORÈME 1. *Pour toute application linéaire et majorée* U *de* $\mathfrak{K}_{\mathfrak{a}}(T)$ *dans* X, *il existe une mesure vectorielle régulière* $\mathbf{m}$ *à variation finie, et une seule, telle que*

$$U(\mathbf{x}) = \int \mathbf{x}\, d\mathbf{m} \qquad \text{pour } \mathbf{x} \in \mathfrak{K}_{\mathfrak{a}}(T).$$

En outre, la variation μ *de* $\mathbf{m}$ *est la plus petitite mesure positive majorante de* U, *et pour tout ensemble ouvert* $G \in \mathfrak{B}$, *on a* $\|U_G\| = \tilde{\mu}(G)$, *où* U_G *est la restriction de* U *à l'espace* $\mathfrak{K}_{\mathfrak{a}}(T, G)$.

La démonstration est la même que pour le cas où T est compact **(8)**.

II. Représentation intégrale des opérations linéaires sur $L_{\mathfrak{a}}^{p}$

Dans la suite de cet article on supposera que:

(1) La famille fondamentale $\mathfrak{a}$ vérifie l'axiome (G), c'est-à-dire qu'il existe une suite $(\mathbf{x}_n)$ de champs de vecteurs appartenant à $\mathfrak{a}$, telle que la suite $(\mathbf{x}_n(t))$ soit dense dans $E(t)$ quel que soit $t \in T$;

(2) X est le dual F' d'un espace de Banach F de type dénombrable.

Les mesures vectorielles seront donc considérées à valeurs dans $\mathfrak{L}(\mathfrak{a}, F')$.

Pour chaque $t \in T$, désignons par $G(t)$ l'espace $\mathfrak{L}(E(t), F')$ des applications linéaires et continues de $E(t)$ dans F', par $\mathfrak{G}$ la famille $(G(t))_{t\in T}$, et par $\mathfrak{C}(\mathfrak{G})$ l'ensemble des champs d'opérations U définis sur T tels que $U(t) \in G(t)$ quel que soit $t \in T$.

1. Mesures vectorielles absolument continues par rapport à une mesure positive. On démontre d'abord le théorème suivant qui est fondamental pour ce qui suit.

Théorème 2. *Pour toute mesure vectorielle régulière* $\mathbf{m}$, *à variation finie* μ, *il existe un champ d'opérations* $U_{\mathbf{m}} \in \mathfrak{C}(\mathfrak{G})$ *déterminé localement* μ*-presque partout, tel que* $\|U_{\mathbf{m}}(t)\| = 1$ *et*

$$\left\langle z, \int \mathbf{x}\, d\mathbf{m} \right\rangle = \int \langle z, U_{\mathbf{m}}(t)\mathbf{x}(t)\rangle\, d\mu(t)$$

quels que soient $z \in F$ *et* $\mathbf{x} \in L_{\mathfrak{a}}^{1}(\mu)$.

Démonstration. Pour tout $z \in F$ et $\mathbf{x} \in \mathfrak{a}$, la fonction d'ensemble $m_{z,\mathbf{x}}$ définie par l'égalité

$$m_{z,\mathbf{x}}(A) = \langle z, \mathbf{m}(A)\mathbf{x}\rangle \qquad \text{pour } A \in \mathfrak{B}$$

$m_{z,\mathbf{x}}$ est une mesure complexe régulière sur T. On a

$$|m_{z,\mathbf{x}}(A)| \leqslant |z|\, \|\mathbf{x}\|_A \mu(A) \qquad \text{pour } A \in \mathfrak{B} \tag{1}$$

donc $m_{z,\mathbf{x}}$ est de base μ. Il existe alors **(2)** une fonction complexe $g_{z,\mathbf{x}}$ définie sur T, localement μ-intégrable (déterminée localement μ-presque partout), telle que

$$m_{z,\mathbf{x}} = g_{z,\mathbf{x}}\mu.$$

Si l'on désigne par $\mu_{z,\mathbf{x}}$ le module (ou la variation) de $m_{z,\mathbf{x}}$ alors une fonction complexe φ définie sur T est essentiellement $\mu_{z,\mathbf{x}}$-intégrable si et seulement si $\varphi g_{z,\mathbf{x}}$ est essentiellement μ-intégrable **(2)** et on a[3]

$$\int \varphi\, dm_{z,\mathbf{x}} = \int \varphi\, g_{z,\mathbf{x}}\, d\mu \qquad \text{pour} \qquad \varphi \in L_c^1(m_{z,\mathbf{x}}). \tag{2}$$

[3] L'intégrale essentielle par rapport à μ d'une fonction φ sera notée aussi par $\int \varphi d\mu$ au lieu de $\int \varphi d\mu$.

On déduit aussitôt qu'on a

$$g_{\alpha_1 z_1+\alpha_2 z_2,\beta_1 \mathbf{x}_1+\beta_2 \mathbf{x}_2} = \alpha_1\beta_1 g_{z_1,\mathbf{x}_1} + \alpha_1\beta_2 g_{z_1,\mathbf{x}_2} + \alpha_2\beta_1 g_{z_2,\mathbf{x}_1} + \alpha_2\beta_2 g_{z_2,\mathbf{x}_2} \tag{3}$$

localement μ-presque partout pour chaques $z_1, z_2 \in F$, $\mathbf{x}_1, \mathbf{x}_2 \in \mathfrak{a}$ et α_1, α_2, $\beta_2, \beta_2 \in C$.

Si $z \in F$, $A \in B$, $\mathbf{x}, \mathbf{y} \in \mathfrak{a}$ et si $\varphi_A \mathbf{x} = \varphi_A \mathbf{y}$, alors

$$g_{z,\mathbf{x}} = g_{z,\mathbf{y}} \tag{4}$$

μ-presque partout sur A.

De (1) on déduit

$$\mu_{z,\mathbf{x}}(A) \leqslant |z|\, ||\mathbf{x}||_A \mu(A) \qquad \text{pour } A \in \mathfrak{B}(\mu)$$

donc aussi

$$\mu_{z,\mathbf{x}}(B) \leqslant |z|\, ||\mathbf{x}||_A \mu(B)$$

quel que soit $B \subset A$, $(B \in \mathfrak{B}(\mu))$. Il s'ensuit que $L^1(\mu_{z,\mathbf{x}}) \supset L^1(\mu)$.

De (2) on déduit alors

$$\left| \int_B \varphi\, g_{z,\mathbf{x}}\, d\mu \right| \leqslant \int_B |\varphi|\, d\mu_{z,\mathbf{x}} \leqslant |z|\, ||\mathbf{x}||_A \int_B |\varphi|\, d\mu \tag{5}$$

quels que soient $B \subset A$, $(A, B \in \mathfrak{B}(\mu))$ et φ localement μ-intégrable. En particulier, comme $\varphi = \overline{g_{z,\mathbf{x}}}$ est localement μ-intégrable, $|g_{z,\mathbf{x}}|^2$ est localement μ-intégrable, et de (5) on déduit:

$$|g_{z,\mathbf{x}}(t)| \leqslant |z|\, ||\mathbf{x}||_A \tag{6}$$

μ-presque partout sur $A \in \mathfrak{B}(\mu)$.

Soit $(\mathbf{x}_n)$ une suite de champs de vecteurs de $\mathfrak{a}$, telle que la suite $(\mathbf{x}_n(t))$ soit dense dans $E(t)$ quel que soit $t \in T$, et soit $\mathfrak{a}_0$ l'ensemble des combinaisons linéaires à coefficients rationnels complexes des éléments de la suite $(\mathbf{x}_n)$. Pour chaque $t \in T$, soit $E_0(t) = \{\mathbf{x}(t) | \mathbf{x} \in \mathfrak{a}_0\}$; $E_0(t)$ est un sous-espace vectoriel (par rapport au corps des nombres rationnels complexes) dense dans $E(t)$.

Soit F_0 le sous-espace de F formé des combinaisons linéaires à coefficients rationnels complexes des éléments d'une partie dénombrable dense dans F.

Il existe un ensemble localement μ-négligeable N_1, tel que pour $t \notin N_1$, les relations (3) soient vérifiées quels que soient $z_1, z_2 \in F_0$, $\mathbf{x}_1, \mathbf{x}_2 \in \mathfrak{a}_0$ et $\alpha_1, \alpha_2, \beta_1, \beta_2$ rationnels complexes.

Il existe un ensemble localement μ-négligeable N_2, tel que pour $t \notin N_2$ on ait

$$|g_{z,\mathbf{x}}(t)| \leqslant |z|\, |\mathbf{x}(t)| \tag{7}$$

quels que soient $z \in F_0$ et $\mathbf{x} \in \mathfrak{a}_0$. En effet, fixons $z \in F_0$ et $\mathbf{x} \in \mathfrak{a}_0$ et posons

$$A(z, \mathbf{x}) = \{t | |g_{z,\mathbf{x}}(t)| > |z|\, |\mathbf{x}(t)|\}.$$

$A(z, \mathbf{x})$ est μ-mesurable. Soit $K \subset A(z, \mathbf{x})$ un ensemble compact, tel que $g_{z,\mathbf{x}}$ soit continue sur K. Il existe un recouvrement fini de K formé par des ensembles $G_1, \ldots, G_n$ ouverts dans K, tels que pour chaque $1 \leqslant i \leqslant n$ on ait

$$\inf_{t \in G_i} |g_{z,\mathbf{x}}(t)| > |z| \sup_{t \in G_i} |\mathbf{x}(t)|.$$

De (6) on déduit que G_t est μ-négligeable, donc K est μ-négligeable. Si maintenant H est un ensemble compact de T, $A(z, \mathbf{x}) \cap H$ est réunion d'un ensemble μ-négligeable et d'une suite (K_n) d'ensembles compacts telle que la restriction de $g_{z,\mathbf{x}}$ sur chaque K_n soit continue. Alors, chaque K_n est μ-négligeable donc $A(z, \mathbf{x}) \cap H$ l'est aussi. Comme H est arbitraire, il résulte que $A(z, \mathbf{x})$ est localement μ-négligeable. On prend alors

$$N_2 = \bigcup_{\substack{z \in F_0 \\ \mathbf{x} \in \mathfrak{a}_0}} A(z, \mathbf{x}),$$

et pour $t \notin N_2$, la relation (7) est vérifiée quels que soient $z \in F_0$ et $\mathbf{x} \in \mathfrak{a}_0$.

Il existe un ensemble localement μ-négligeable N_3, tel que pour chaque $t \notin N_3$ on ait

$$g_{z,\mathbf{x}}(t) = g_{z,\mathbf{y}}(t) \tag{8}$$

quels que soient $z \in F_0$ et $\mathbf{x}, \mathbf{y} \in \mathfrak{a}_0$ vérifiant $\mathbf{x}(t) = \mathbf{y}(t)$.

En effet, fixons $z \in F_0$ et $\mathbf{x}, \mathbf{y} \in \mathfrak{a}_0$ et posons

$$B(z, \mathbf{x}, \mathbf{y}) = \{t | \mathbf{x}(t) = \mathbf{y}(t), \quad g_{z,\mathbf{x}}(t) \neq g_{z,\mathbf{y}}(t)\}.$$

$B(z, \mathbf{x}, \mathbf{y})$ est μ-mesurable. Si $K \subset T$ est un compact quelconque, de (4) on déduit que $K \cap B(z, \mathbf{x}, \mathbf{y})$ est μ-négligeable, donc $B(z, \mathbf{x}, \mathbf{y})$ est localement μ-négligeable. On prend alors N_3 la réunion des $B(z, \mathbf{x}, \mathbf{y})$, $(z \in F_0, \mathbf{x}, \mathbf{y} \in \mathfrak{a}_0)$.

Soit

$$N = \bigcup_{i=1}^{3} N_i.$$

Si pour tout $t \in N$ on pose $g_{z,\mathbf{x}}(t) = 0$ quels que soient $z \in F_0$ et $\mathbf{x} \in \mathfrak{a}_0$, alors les relations (3), (7) et (8) sont vérifiées pour tout $t \in T$, quels que soient les indices appartenant à F_0 et $\mathfrak{a}_0$, et les nombres rationnels complexes α_1, α_2, β_1, β_2.

Fixons maintenant $t \in T$ et $z \in F_0$. Pour chaque $a \in E_0(t)$ on pose

$$g_{z,a}(t) = g_{z,\mathbf{x}}(t)$$

où $\mathbf{x} \in \mathfrak{a}_0$ est tel que $\mathbf{x}(t) = a$. De l'égalité (8) on déduit que $g_{z,a}(t)$ ne dépend que de $a \in E_0(t)$.

Pour $t \in T$ et $a \in E_0(t)$ fixes, l'application $z \to g_{z,a}(t)$ de F_0 dans C est linéaire (par rapport au corps des nombres rationnels complexes) et continue (7), donc peut être prolongée uniquement par continuité à une forme linéaire et continue $g_a(t)$ sur F (donc $g_a(t) \in F'$). De (7) on déduit qu'on a

$$|g_a(t)| \leqslant |a|. \tag{9}$$

Pour tout $z \in F_0$ on a

$$\langle z, g_a(t) \rangle = g_{z,a}(t).$$

Fixons maintenant $t \in T$. L'application $a \to g_a(t)$ de $E_0(t)$ dans F' est linéaire (par rapport au corps des nombres rationnels complexes) et continue, donc

peut être prolongée à une application linéaire et continue $U_{\mathbf{m}}(t)$ de $E(t)$ dans F' (donc $U_{\mathbf{m}}(t) \in \mathfrak{L}(E(t), F')$) et

$$\|U_{\mathbf{m}}(t)\| \leqslant 1. \tag{10}$$

Pour $\mathbf{x} \in \mathfrak{a}_0$ on a donc

$$U_{\mathbf{m}}(t)\mathbf{x}(t) = g_{\mathbf{x}(t)}(t)$$

quel que soit $t \in T$, donc pour $z \in F_0$, $\mathbf{x} \in \mathfrak{a}_0$ et $t \in T$ on a

$$\langle z, U_{\mathbf{m}}(t)\mathbf{x}(t)\rangle = g_{z,\mathbf{x}}(t)$$

d'où il résulte, pour $A \in \mathfrak{B}$,

$$\left\langle z, \int \varphi_A \mathbf{x}\, d\mathbf{m} \right\rangle = \int \langle z, U_{\mathbf{m}}(t)\varphi_A(t)\mathbf{x}(t)\rangle\, d\mu(t). \tag{11}$$

Si

$$\mathbf{x} = \sum_i \varphi_{A_i} \cdot \mathbf{x}_i,\ (A_i \in \mathfrak{B}, \mathbf{x}_i \in \mathfrak{a}_0)$$

et $z \in F_0$, on déduit

$$\left\langle z, \int \mathbf{x}\, d\mathbf{m} \right\rangle = \int \langle z, U_{\mathbf{m}}(t)\mathbf{x}(t)\rangle\, d\mu(t). \tag{12}$$

Si $\mathbf{x} \in \mathfrak{a}$, et $A \in \mathfrak{B}$ il existe une suite $(\mathbf{x}_n)$ de champs de vecteurs de la forme précédente, uniformément convergente vers $\varphi_A \mathbf{x}$; alors, pour $z \in F_0$ la suite $\langle z, U_{\mathbf{m}}(t)\mathbf{x}_n(t)\rangle$ est uniformément convergente vers $\langle z, U_{\mathbf{m}}(t)\varphi_A(t)\mathbf{x}(t)\rangle$, donc l'égalité (11) est vraie pour $\mathbf{x} \in \mathfrak{a}$ et $A \in \mathfrak{B}$. Alors l'égalité (12) est vraie pour tout champ étagé $\mathbf{x} \in \mathfrak{E}_{\mathfrak{a}}(T)$, et en appliquant le théorème de Lebesgue on déduit qu'elle est vraie pour tout $\mathbf{x} \in L_{\mathfrak{a}}^1(\mathbf{m})$ et tout $z \in F$.

Montrons maintenant que $\|U_{\mathbf{m}}(t)| \geqslant 1$ localement μ-presque partout, d'où il s'ensuit que $\|U_{\mathbf{m}}(t)| \equiv 1$ localement μ-presque partout, donc partout si l'on modifie $U_{\mathbf{m}}$ sur un ensemble localement μ-négligeable.

Pour chaque $\mathbf{x} \in \mathfrak{a}_0$, la fonction $t \to |U_{\mathbf{m}}(t)\mathbf{x}(t)|$ est μ-mesurable, comme borne supérieure de la suite de fonctions mesurables $t \to |\langle z_n, U_{\mathbf{m}}(t)\mathbf{x}(t)\rangle|$ où (z_n) est une suite dense dans la sphère unité de F.

La fonction $t \to \|U_{\mathbf{m}}(t)\|$ est aussi μ-mesurable, comme borne supérieure de la suite

$$t \to \frac{|U_{\mathbf{m}}(t)\mathbf{x}(t)|}{|\mathbf{x}(t)|},\ \mathbf{x} \in \mathfrak{a}_0$$

où l'on convient de poser

$$\frac{|U_{\mathbf{m}}(t)\mathbf{x}(t)|}{|\mathbf{x}(t)|} = 0$$

si $\mathbf{x}(t) = 0$.

Pour chaque nombre naturel n posons

$$B_n = \left\{t \mid \|U_{\mathbf{m}}(t)\| < 1 - \frac{1}{n}\right\}.$$

B_n est μ-mesurable, donc est contenu dans un ensemble $B_n' \in \mathfrak{B}'$ tel que $B_n' - B_n$ soit localement μ-négligeable.

Soit $K \subset T$ un ensemble compact. Pour tout $z \in F$, $\mathbf{x} \in \mathfrak{a}$ et $A \in \mathfrak{B}$ on a

$$|\langle z, \mathbf{m}(K \cap B_n' \cap A)\mathbf{x}\rangle| = \left|\int_{K \cap B_n' \cap A} \langle z, U_{\mathbf{m}}(t)\mathbf{x}(t)\rangle \, d\mu(t)\right|$$

$$< \left(1 - \frac{1}{n}\right) |z| \, ||\mathbf{x}||_{K \cap B_n'} \mu(K \cap B_n' \cap A).$$

On déduit alors

$$||\mathbf{m}(K \cap B_n' \cap A)|| < \left(1 - \frac{1}{n}\right) \mu(K \cap B_n' \cap A)$$

donc si (A_ι) est une partition finie quelconque de $K \cap B_n'$ en ensembles de $\mathfrak{B}$, on a

$$\sum_\iota ||\mathbf{m}(K \cap B_n' \cap A_\iota)|| < \left(1 - \frac{1}{n}\right) \mu(K \cap B_n')$$

donc

$$\mu(K \cap B_n') < \left(1 - \frac{1}{n}\right) \mu(K \cap B_n')$$

ce qui entraîne que $K \cap B_n'$ est μ-négligeable, donc B_n' (et aussi B_n) est localement μ-négligeable.

La réunion

$$B = \bigcup_{n=1}^{\infty} B_n$$

est localement μ-négligeable.

Mais

$$B = \{t \mid ||U_{\mathbf{m}}(t)|| < 1\}$$

donc on a $||U_{\mathbf{m}}(t)|| \geqslant 1$ localement μ-presque partout.

Il reste à montrer l'unicité de $U_{\mathbf{m}}$. Soit V un champ d'opérations défini sur T tel que $V(t) \in \mathfrak{L}(E(t), F')$ quel que soit $t \in T$, et tel en outre que

$$\left\langle z, \int \mathbf{x} \, d\mathbf{m} \right\rangle = \int \langle z, V(t)\mathbf{x}(t)\rangle \, d\mu(t)$$

quels que soient $z \in F$ et $\mathbf{x} \in L_{\mathfrak{a}}^1(\mathbf{m})$. On déduit alors que

$$\int_A \langle z, (U_{\mathbf{m}}(t) - V(t))\mathbf{x}(t)\rangle \, d\mu(t) = 0$$

quels que soient $z \in F$, $A \in \mathfrak{B}$ et $\mathbf{x} \in \mathfrak{a}$, donc

$$\langle z, (U_{\mathbf{m}}(t) - V(t))\mathbf{x}(t)\rangle = 0$$

localement μ-presque partout, pour chaque $z \in F$ et $\mathbf{x} \in \mathfrak{a}$. En faisant z parcourir F_0 on déduit d'abord

$$U_{\mathfrak{m}}(t)\mathbf{x}(t) = V(t)\mathbf{x}(t)$$

localement μ-presque partout pour chaque $\mathbf{x} \in \mathfrak{a}$.

En faisant $\mathbf{x}$ parcourir $\mathfrak{a}_0$, on déduit enfin que

$$U_{\mathfrak{m}}(t) = V(t)$$

localement μ-presque partout. Ceci achève la démonstration du théorème.

Remarques.

1°. Soit $U \in \mathfrak{L}(\mathfrak{G})$ tel que pour chaque $z \in F$ et $\mathbf{x} \in \mathfrak{a}$, la fonction $t \to \langle z, U(t)\mathbf{x}(t)\rangle$ soit μ-mesurable. Alors, pour tout $z \in F$ et tout champ $\mathbf{x} \in \mathfrak{L}(\mathfrak{E})$ mesurable par rapport à μ et $\mathfrak{a}$, les fonctions $t \to \langle z, U(t)\mathbf{x}(t)\rangle$, $t \to |U(t)\mathbf{x}(t)|$ et $t \to \|U(t)\|$ sont μ-mesurables.

En effet, soit $z \in F$ et $\mathbf{x} \in \mathfrak{L}(\mathfrak{E})$ mesurable par rapport à μ et $\mathfrak{a}$. Si $K \subset T$ est compact, alors $\varphi_K\mathbf{x}$ est limite μ-presque partout d'une suite $(\mathbf{x}_n)$ de champs étagés, donc $t \to \langle z, U(t)\mathbf{x}(t)\rangle\varphi_K(t)$ est μ-mesurable comme limite μ-presque partout de la suite de fonctions μ-mesurables $t \to \langle z, U(t)\mathbf{x}_n(t)\rangle$. On applique ensuite le principe de la localisation et on déduit que $t \to \langle z, U(t)\mathbf{x}(t)\rangle$ est μ-mesurable.

Puisque $t \to |\langle z, U(t)\mathbf{x}(t)\rangle|$ est μ-mesurable, en faisant z parcourir une suite dense dans la sphère unité de F, on déduit que $t \to |U(t)\mathbf{x}(t)|$ est μ-mesurable. Enfin, la fonction $t \to \|U(t)\|$ est μ-mesurable comme borne supérieure de la suite de fonctions μ-mesurables

$$t \to \frac{|U(t)\mathbf{x}(t)|}{|\mathbf{x}(t)|}, \mathbf{x} \in \mathfrak{a}_0,$$

où l'on pose

$$\frac{|U(t)\mathbf{x}(t)|}{|\mathbf{x}(t)|} = 0$$

si $\mathbf{x}(t) = 0$.

En particulier, cette remarque est valable pour $U_{\mathfrak{m}}$.

2°. Si F' est de type dénombrable, alors pour tout champ $\mathbf{x} \in \mathfrak{L}(\mathfrak{E})$ mesurable par rapport à μ et $\mathfrak{a}$, la fonction $t \to U_{\mathfrak{m}}(t)\mathbf{x}(t)$ est μ-mesurable.

En effet, pour chaque $z \in F$ et $z' \in F'$ la fonction $t \to \langle z, U_{\mathfrak{m}}(t)\mathbf{x}(t)\rangle - \langle z, z'\rangle$ est μ-mesurable. En faisant z parcourir une suite dense dans la sphère unité de F, on déduit que la fonction $t \to |U_{\mathfrak{m}}(t)\mathbf{x}(t) - z'|$ est μ-mesurable. Soit (z_n') une suite dense dans F', $\epsilon > 0$ et $K \subset T$ compact. Pour chaque n posons

$$B_n = \{t \mid |U_{\mathfrak{m}}(t)\mathbf{x}(t) - z_n'| < \epsilon\} \cap K.$$

Alors B_n est μ-mesurable et

$$\bigcup_{n=1}^{\infty} B_n = K.$$

Il existe une suite (A_n) d'ensembles disjoints μ-mesurables, telle que

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$$

et $A_n \subset B_n$ pour chaque n. Posons

$$\mathbf{z}_\epsilon' = \sum_{n=1}^{\infty} \varphi_{A_n} \mathbf{z}_n'.$$

$\mathbf{z}_\epsilon'$ est μ-mesurable et pour chaque $t \in K$ on a

$$|U_{\mathbf{m}}(t)\mathbf{x}(t) - \mathbf{z}_\epsilon'(t)| \leqslant \epsilon.$$

Il s'ensuit que $t \to U_{\mathbf{m}}(t)\mathbf{x}(t)$ est μ-mesurable sur K, comme limite uniforme sur K, de fonctions μ-mesurables. En vertu du principe de la localisation, $t \to U_{\mathbf{m}}(t)\mathbf{x}(t)$ est μ-mesurable.

Comme $|U_{\mathbf{m}}(t)\mathbf{x}(t)| \leqslant \|U_{\mathbf{m}}(t)\|\,|\mathbf{x}(t)| = |\mathbf{x}(t)|$, il s'ensuit que si $\mathbf{x} \in L_{\mathfrak{a}}^1(\mathbf{m})$, alors la fonction $t \to U_{\mathbf{m}}(t)\mathbf{x}(t)$ est μ-intégrable, donc:

Si F' est de type dénombrable, alors pour chaque $\mathbf{x} \in L_{\mathfrak{a}}^1(\mathbf{m})$ on a

$$\int \mathbf{x}\, d\mathbf{m} = \int U_{\mathbf{m}}(t)\mathbf{x}(t)\, d\mu(t)$$

et pour chaque $\mathbf{x} \in \mathfrak{a}$ et $A \in \mathfrak{B}$ on a

$$\mathbf{m}(A)\mathbf{x} = \int_A U_{\mathbf{m}}(t)\mathbf{x}(t)\, d\mu(t).$$

Dans ce cas, le théorème 2 a été démontré par une voie différente dans **(8)**.

3°. S'il existe une famille fondamentale $\mathfrak{D} \subset \mathfrak{C}(\mathfrak{G})$ vérifiant l'axiome (G) et la condition suivante:

(T) la fonction $t \to V(t)\mathbf{x}(t)$ est μ-mesurable quel que soit $V \in \mathfrak{D}$ et $\mathbf{x} \in \mathfrak{a}$, alors $U_{\mathbf{m}}$ est mesurable, relativement à μ et $\mathfrak{D}$, donc

$$U_{\mathbf{m}} \in L_{\mathfrak{D}}^{\infty}(\mu).$$

En effet, soit (V_n) une suite d'éléments de $\mathfrak{D}$, telle que la suite $(V_n(t))$ soit dense dans $G(t)$ quel que soit $t \in T$. On déduit d'abord que chaque $G(t)$ est de type dénombrable, donc F' l'est aussi; alors $t \to U_{\mathbf{m}}(t)\mathbf{x}(t)$ est μ-mesurable pour tout $\mathbf{x} \in \mathfrak{a}$ (remarque 2). Pour chaque n, la fonction $t \to V_n(t)\mathbf{x}(t) - U_{\mathbf{m}}(t)\mathbf{x}(t)$ est alors μ-mesurable donc la fonction $t \to \|V_n(t) - U_{\mathbf{m}}(t)\|$ est μ-mesurable (remarque 1).

Soit $\epsilon > 0$ et $K \subset T$ compact. Pour chaque n posons

$$B_n = \{t \mid \|V_n(t) - U_{\mathbf{m}}(t)\| \leqslant \epsilon\} \cap K.$$

B_n est μ-mesurable et

$$\bigcup_{n=1}^{\infty} B_n = K.$$

Soit (A_n) une suite d'ensembles disjoints μ-mesurables, telle que

$$\bigcup_{n=1}^{\infty} A_n = K$$

et $A_n \subset B_n$ pour chaque n. Posons

$$U_\epsilon = \sum_{n=1}^{\infty} \varphi_{A_n} V_n.$$

U_ε est mesurable par rapport à μ et $\mathfrak{D}$, et pour $t \in K$ on a

$$\|U_\varepsilon(t) - U_{\mathbf{m}}(t)\| \leqslant \varepsilon.$$

Il s'ensuit que $U_{\mathbf{m}}$ est mesurable (par rapport à μ et $\mathfrak{D}$) sur K, comme limite uniforme sur K de champs d'opérations mesurables, donc, en vertu du principe de la localisation, $U_{\mathbf{m}}$ est mesurable.

4°. Si $E(t) = E$ quel que soit $t \in T$, et $\mathfrak{L}(E, F')$ est de type dénombrable, alors, pour chaque $A \in \mathfrak{B}$ on a

$$\mathbf{m}(A) = \int_A U_{\mathbf{m}}(t)\, d\mu(t).$$

5°. Si $F = C$, alors $G(t) = E'(t)$ pour chaque $t \in T$. Dans ce cas, $\mathbf{x}_{\mathbf{m}}' = U_{\mathbf{m}}$ est un champ de fonctionnelles de $\mathfrak{C}(\mathfrak{E}')$, où $\mathfrak{E}' = (E'(t))_{t \in T}$. On peut alors écrire

$$\int \mathbf{x}\, d\mathbf{m} = \int \langle \mathbf{x}(t), \mathbf{x}_{\mathbf{m}}'(t) \rangle\, d\mu(t) \qquad \text{pour } \mathbf{x} \in L_{\mathfrak{a}}^1(\mathbf{m}).$$

S'il existe une famille $\mathfrak{a}' \subset \mathfrak{C}(\mathfrak{E}')$, vérifiant l'axiome (G) et telle que la fonction scalaire $t \to \langle \mathbf{x}(t), \mathbf{x}'(t) \rangle$ soit μ-mesurable quels que soient $\mathbf{x} \in \mathfrak{a}$ et $\mathbf{x}' \in \mathfrak{a}'$, alors $\mathbf{x}_{\mathbf{m}}'$ est μ-mesurable, donc

$$\mathbf{x}_{\mathbf{m}}' \in L_{\mathfrak{a}'}^{\infty}(\mu).$$

6°. Si $E(t) = C$ quel que soit $t \in T$, alors $\mathfrak{L}(C, F')$ peut être identifié à F'. Dans ce cas $U_{\mathbf{m}}$ est une application de T dans F', donc

$$\left\langle z, \int \varphi(t) d\mathbf{m}(t) \right\rangle = \int \varphi(t)\, \langle z, U_{\mathbf{m}}(t) \rangle\, d\mu(t) \qquad \text{pour } \varphi \in L_C^1(\mu), z \in F.$$

Si F' est de type dénombrable, $U_{\mathbf{m}}$ est μ-mesurable et on a

$$\int \varphi(t) d\mathbf{m}(t) = \int \varphi(t) U_{\mathbf{m}}(t)\, d\mu(t) \qquad \text{pour } \varphi \in L_C^1(\mu)$$

et

$$\mathbf{m}(A) = \int_A U_{\mathbf{m}}(t)\, d\mu(t) \qquad \text{pour } A \in \mathfrak{B}.$$

7°. Si m est une mesure complexe, alors $U_{\mathbf{m}}$ est une fonction complexe sur T, μ-mesurable, et à valeurs sur le cercle unité du plan complexe.

8°. Si m est une mesure réelle, alors $U_{\mathbf{m}}$ est à valeurs dans l'ensemble $\{-1, 1\}$, donc est de la forme $U_{\mathbf{m}} = \varphi_A - \varphi_B$, où $A \cup B = T$ et $A \cap B = \phi$. Dans ce cas

$$m = \varphi_A \mu - \varphi_B \mu, \qquad \mu = \varphi_A \mu + \varphi_B \mu$$

donc $m^+ = \varphi_A \mu$ et $m^- = \varphi_B \mu$.

Soit ν une mesure positive sur T ($\nu(A) < \infty$ pour $A \in \mathfrak{B}$).

On dit qu'une mesure vectorielle $\mathbf{m}$ est absolument continue par rapport à ν, si $\nu(A) = 0$ implique $\mathbf{m}(A) = 0$.

PROPOSITION 9. *Soit ν une mesure positive sur T. Une mesure vectorielle* $\mathbf{m}$, *à variation finie μ est absolument continue par rapport à ν, si et seulement si sa variation μ est absolument continue par rapport à ν.*

Soit $A \in \mathfrak{B}$ telle que $\nu(A) = 0$. Si μ est absolument continue par rapport à ν, on a $\mu(A) = 0$ donc $\mathbf{m}(A) = 0$, c'est-à-dire, $\mathbf{m}$ est absolument continue par rapport à ν. Inversement, si $\mathbf{m}$ est absolument continue par rapport à ν, pour tout $B \in \mathfrak{B}$ tel que $B \subset A$ on a $\mathbf{m}(B) = 0$ (car $\nu(B) = 0$), donc $\mu(A) = 0$, c'est-à-dire μ est absolument continue par rapport à ν.

Remarque. Si ν est régulière et $\mathbf{m}$ est absolument continue par rapport à ν, alors $\mathbf{m}$ est aussi régulière (car sa variation μ est régulière).

Le théorème suivant est une généralisation du théorème bien connu de Lebesgue–Nikodym.

THÉORÈME 3. *Soit ν une mesure régulière positive sur T. Une mesure vectorielle* $\mathbf{m}$ *est à variation finie et absolument continue par rapport à ν, si et seulement si il existe un champ d'opérations $V_{\mathbf{m}} \in \mathfrak{C}(\mathfrak{G})$, tel que la fonction $t \to \|V_{\mathbf{m}}(t)\|$ soit localement ν-intégrale et que pour tout $z \in F$ et $\mathbf{x} \in \mathfrak{L}_s^1(\mathbf{m})$ on ait*

$$\left\langle z, \int \mathbf{x} d\mathbf{m} \right\rangle = \int \langle z, V_{\mathbf{m}}(t)\mathbf{x}(t)\rangle \, d\nu(t).$$

Le champ d'opérations $V_{\mathbf{m}}$ est alors déterminé localement ν-presque partout, et pour chaque $A \in \mathfrak{B}$ on a

$$\mu(A) = \int_A \|V_{\mathbf{m}}(t)\| \, d\nu(t).$$

Démonstration. Supposons que $\mathbf{m}$ est à variation finie et absolument continue par rapport à ν. Alors sa variation μ l'est aussi, donc il existe une fonction positive g sur T, localement ν-intégrable (déterminée localement ν-presque partout), telle qu'une fonction complexe φ est essentiellement μ-intégrable si et seulement si φg est essentiellement ν-intégrable; on a alors

$$\int \varphi \, d\mu = \int \varphi g \, d\nu \qquad \text{pour } \varphi \in L_C^1(\mu).$$

D'après le théorème 2, il existe un champ d'opérations $U_{\mathbf{m}} \in \mathfrak{C}(\mathfrak{G})$ (déterminé localement μ-presque partout), tel que $\|U_{\mathbf{m}}(t)\| \equiv 1$ et que pour tout $\mathbf{x} \in \mathfrak{L}_s^1(\mathbf{m})$ et $z \in F$ on ait $\langle z, U_{\mathbf{m}}\mathbf{x}\rangle \in L_C^1(\mu)$ et

$$\left\langle z, \int \mathbf{x} \, d\mathbf{m} \right\rangle = \int \langle z, U_{\mathbf{m}}(t)\mathbf{x}(t)\rangle \, d\mu(t).$$

Si l'on pose $V_{\mathbf{m}} = gU_{\mathbf{m}}$ alors $\|V_{\mathbf{m}}(t)\| = g(t)$, donc la fonction $t \to \|V_{\mathbf{m}}(t)\|$ est localement ν-intégrable et

$$\mu(A) = \int_A \|V_{\mathbf{m}}(t)\| \, d\nu(t)$$

pour chaque $A \in \mathfrak{B}$. Si $\mathbf{x} \in L_{\mathfrak{a}}^1(\mathbf{m})$ alors la fonction $\langle z, V_{\mathbf{m}}\mathbf{x}\rangle$ est essentiellement ν-intégrable quel que soit $z \in F$ et on a

$$\left\langle z, \int \mathbf{x}\, d\mathbf{m} \right\rangle = \int \langle z, V_{\mathbf{m}}(t)\mathbf{x}(t)\rangle\, d\nu(t).$$

L'unicité de $V_{\mathbf{m}}$ se déduit comme dans la démonstration du théorème 2.

Inversement, si V est un champ d'opérations tel que la fonction $t \to \|V(t)\|$ soit localement ν-intégrable et que pour tout $z \in F$ et $\mathbf{x} \in L_{\mathfrak{a}}^1(\mathbf{m})$ on ait

$$\left\langle z, \int \mathbf{x}\, d\mathbf{m} \right\rangle = \int \langle z, V(t)\mathbf{x}(t)\rangle\, d\nu(t)$$

alors $\mathbf{m}$ est absolument continue par rapport à ν. On a alors

$$\|\mathbf{m}(A)\| \leqslant \int_A \|V(t)\, d\nu(t) \qquad \text{pour } A \in \mathfrak{B}$$

d'où l'on déduit que $\mathbf{m}$ est à variation finie μ. De la première partie de la démonstration on déduit que $V = V_{\mathbf{m}}$ localement ν-presque partout.

Remarques.

1°. Si F' est de type dénombrable, alors

$$\int \mathbf{x}\, d\mathbf{m} = \int V_{\mathbf{m}}(t)\mathbf{x}(t) d\nu(t) \qquad \text{pour } \mathbf{x} \in L_{\mathfrak{a}}^1(\mathbf{m})$$

$$\mathbf{m}(A)\mathbf{x} = \int_A V_{\mathbf{m}}(t)\mathbf{x}(t) d\nu(t) \qquad \text{pour } A \in \mathfrak{B} \text{ et } \mathbf{x} \in \mathfrak{a}.$$

2°. Si $E(t) = E$ quel que soit $t \in T$ et si $\mathfrak{L}(E, F')$ est de type dénombrable, alors

$$\mathbf{m}(A) = \int_A V_{\mathbf{m}}(t) d\nu(t) \qquad \text{pour } A \in \mathfrak{B}.$$

3°. Si l'espace F ou si les espaces $E(t)$ ne sont pas de type dénombrable, il est possible que $V_{\mathbf{m}}$ ne soit pas uniquement déterminé (à un ensemble localement ν-négligeable près).

Exemple 1. $T = [0, 1]$; ν la mesure de Lebesgue sur T; $E(t) = E = C(T)$ quel que soit $t \in T$, où $C(T)$ est l'espace des fonctions numériques continues sur T, muni de la topologie définie par la norme

$$\|f\| = \sup_{t \in T} |f(t)|;$$

F l'espace des fonctions numériques z définies sur T, telles que

$$\|z\| = \sum_{t \in T} |z(t)| < +\infty;$$

F' est alors l'espace des fonctions numériques z' définies sur T telles que

$$\|z'\| = \sup_{t \in T} |z'(t)| < +\infty.$$

L'espace E est de type dénombrable, tandis que F n'est pas de type dénombrable. Toute fonction $z \in F$ est nulle en dehors d'un ensemble dénombrable, donc ν-négligeable.

Considérons la mesure $\mathbf{m} = 0$ sur T à valeurs dans $\mathfrak{L}(E, F')$.

Pour chaque $t \in T$ considérons l'application $V(t):E \to F'$ définie par $V(t)f = \varphi_{\{t\}}f$ pour $f \in E$. On a $||V(t)|| = 1$ quel que soit $t \in T$ et

$$\langle z, V(t)f\rangle = \langle z, \varphi_{\{t\}}f\rangle = \sum_{s \in T} z(s)\varphi_{\{t\}}(s)f(s) = z(t)f(t)$$

quels que soient $z \in F$ et $f \in E$. Soit $\mathbf{x} \in L_E{}^1(\mathbf{m})$; pour chaque $t \in T$ posons $\mathbf{x}(t) = f_t \in E$. Alors, pour $z \in F$,

$$\int \langle z, V(t)\mathbf{x}(t)\rangle\, d\nu(t) = \int \langle z, V(t)f_t\rangle\, d\nu(t) = \int z(t)f_t(t)d\nu(t) = 0$$

donc

$$\left\langle z, \int \mathbf{x}\, d\mathbf{m}\right\rangle = \int \langle z, V(t)\mathbf{x}(t)\rangle\, d\nu(t).$$

D'autre coté, pour $U(t) \equiv 0$ on a aussi

$$\left\langle z, \int \mathbf{x}\, d\mathbf{m}\right\rangle = \int \langle z, U(t)\mathbf{x}(t)\rangle\, d\nu(t)$$

et $V(t) \neq U(t)$ quel que soit $t \in T$.

Exemple 2. T et ν comme plus haut; pour tout $t \in T$, $E(t) = \mathrm{E}$ l'espace des fonctions $f: T \to R$ telles que

$$||f|| = \sum_{t \in T} |f(t)| < +\infty;$$

$F = C(T)$, donc F' est l'espace des mesures de Radon sur T. L'espace F est de type dénombrable, mais E n'est pas de type dénombrable.

Pour tout $t \in T$ considérons l'application $V(t): E \to F'$ définie par $V(t)f = f(t)\epsilon_t$ pour $f \in E$, où ϵ_t est la mesure de masse totale 1 concentrée au point t. On a $||V(t)|| = 1$ quel que soit $t \in T$ et $\langle z, V(t)f\rangle = \langle z, f(t)\epsilon_t\rangle = z(t)f(t)$, quels que soient $z \in F$ et $f \in E$. On vérifie comme plus haut que, pour $\mathbf{m} = 0$ et $U(t) \equiv 0$ on a

$$\left\langle z, \int \mathbf{x}\, d\mathbf{m}\right\rangle = \int \langle z, V(t)\mathbf{x}(t)\rangle\, d\nu(t) = \int \langle z, U(t)\mathbf{x}(t)\rangle\, d\nu(t) = 0$$

quels que soient $z \in F$ et $\mathbf{x} \in L_E{}^1(\mathbf{m})$.

Voici maintenant le théorème réciproque du théorème 2.

THÉORÈME 4. *Soit μ une mesure positive régulière sur T et U un champ d'opérations tel que $||U(t)|| \equiv 1$ et que pour tout $z \in F$ et tout $\mathbf{x} \in \mathfrak{a}$, la fonction $t \to \langle z, U(t)\mathbf{x}(t)\rangle$ soit μ-mesurable. Alors la fonction d'ensemble $\mathbf{m}$, définie par l'égalité*

$$\langle z, \mathbf{m}(A)\rangle = \int_A \langle z, U(t)\mathbf{x}(t)\rangle\, d\mu(t)$$

pour $z \in F$, $\mathbf{x} \in \mathfrak{a}$, $A \in \mathfrak{B}$, est une mesure vectorielle sur T, dont la variation est μ.

Démonstration. On déduit d'abord que la fonction $t \to \langle z, U(t)\mathbf{x}(t)\rangle$ est localement μ-intégrable quel que soient $z \in F$ et $\mathbf{x} \in \mathfrak{a}$. Pour $z \in F$, $\mathbf{x} \in \mathfrak{a}$ et $A \in \mathfrak{B}$ posons

$$M(z, \mathbf{x}, A) = \int_A \langle z, U(t)\mathbf{x}(t)\rangle \, d\mu(t).$$

L'application $M(\mathbf{x}, A): z \to M(z, \mathbf{x}, A)$ est une forme linéaire et continue sur F (donc $M(\mathbf{x}, A) \in F'$) car

$$|M(z, \mathbf{x}, A)| \leqslant |z| \, ||\mathbf{x}||_A \mu(A).$$

On a donc $|M(\mathbf{x}, A)| \leqslant ||\mathbf{x}||_A \mu(A)$ et $\langle z, M(\mathbf{x}, A)\rangle = M(z, \mathbf{x}, A)$.

L'application $\mathbf{m}(A): \mathbf{x} \to M(\mathbf{x}, A)$ de $\mathfrak{a}$ dans F' est une application linéaire et continue (donc $\mathbf{m}(A) \in \mathfrak{L}(\mathfrak{a}, F')$ et $\langle z, \mathbf{m}(A)\mathbf{x}\rangle = M(z, \mathbf{x}, A)$, donc

$$\langle z, \mathbf{m}(A)\mathbf{x}\rangle = \int_A \langle z, U(t)\mathbf{x}(t)\rangle \, d\mu(t).$$

On vérifie aisément que si $\varphi_A \mathbf{x} = \varphi_A \mathbf{y}$, $(A \in \mathfrak{B}, \mathbf{x}, \mathbf{y} \in \mathfrak{a})$ alors $\mathbf{m}(A)\mathbf{x} = \mathbf{m}(A)\mathbf{y}$, donc $\mathbf{m}$ est une fonction vectorielle d'ensemble, et $||\mathbf{m}(A)|| \leqslant \mu(A)$ pour $A \in \mathfrak{B}$. De l'inégalité $||\mathbf{m}(A)|| \leqslant \mu(\mathrm{A})$ on déduit que $\mathbf{m}$ est une mesure vectorielle à variation finie λ et que $\lambda \leqslant \mu$.

La mesure vectorielle $\mathbf{m}$ est régulière puisque μ est régulière. En outre, $\mathbf{m}$ est absolument continue par rapport à μ, et du théorème 3 on déduit que

$$\lambda(A) = \int_A ||U(t)|| d\mu(t) = \mu(A)$$

et ceci achève la démonstration.

On dit que deux mesures positives μ et ν sont étrangères, si $\lambda \leqslant \mu$ et $\lambda \leqslant \nu$ implique $\lambda = 0$.

Deux mesures vectorielles $\mathbf{m}$ et $\mathbf{n}$ à variation finie sont étrangères si leurs variations μ et ν sont étrangères.

Le théorème suivant est une généralisation du théorème de Lebesgue.

THÉORÈME 5. *Soit ν une mesure régulière positive sur T. Toute mesure vectorielle régulière et à variation finie $\mathbf{m}$ sur T peut s'écrire sous la forme*

$$\mathbf{m} = \mathbf{n}' + \mathbf{m}'$$

où $\mathbf{n}'$ et $\mathbf{m}'$ sont des mesures vectorielles régulières et à variation finie, telles que $\mathbf{n}'$ soit absolument continue par rapport à ν, et que $\mathbf{m}'$ soit étrangère à ν.

Démonstration. Soit μ la variation de $\mathbf{m}$. D'après le théorème de Lebesgue **(2)**, μ peut s'écrire sous la forme

$$\mu = \nu' + \mu'$$

où ν' et μ' sont des mesures régulières positives, telles que ν' soit absolument continue par rapport à ν et μ' soit étrangère à ν.

D'autre côté, $\mathbf{m}$ peut s'écrire

$$\langle z, \mathbf{m}(A)\mathbf{x}\rangle = \int_A \langle z, U_{\mathbf{m}}(t)\mathbf{x}(t)\rangle\, d\mu(t)$$

pour $z \in F$, $\mathbf{x} \in \mathfrak{a}$, et $A \in \mathfrak{B}$, où $\|U_{\mathbf{m}}(t)\| = 1$. On a alors

$$\langle z, \mathbf{m}(A)\rangle = \int_A \langle z, U_{\mathbf{m}}(t)\mathbf{x}(t)\rangle\, d\nu'(t) + \int_A \langle z, U_{\mathbf{m}}(t)\mathbf{x}(t)\rangle\, d\mu'(t).$$

Si l'on pose

$$\langle z, \mathbf{n}'(A)\mathbf{x}\rangle = \int_A \langle z, U_{\mathbf{m}}(t)\mathbf{x}(t)\rangle\, d\nu'(t)$$

et

$$\langle z, \mathbf{m}'(A)\mathbf{x}\rangle = \int_A \langle z, U_{\mathbf{m}}(t)\mathbf{x}(t)\rangle\, d\mu'(t),$$

alors $\mathbf{n}'$ et $\mathbf{m}'$ sont des mesures vectorielles régulières dont les variations sont respectivement ν' et μ', car pour tout $z \in F$ et $\mathbf{x} \in \mathfrak{a}$, la fonction $t \to \langle z, U_{\mathbf{m}}(t)\mathbf{x}(t)\rangle$ est mesurable aussi parr apport à ν' et par rapport à μ'. En outre, $\mathbf{n}'$ est absolument continue par rapport à ν, et $\mathbf{m}'$ est étrangère à ν, et pour tout $A \in \mathfrak{B}$ on a

$$\mathbf{m}(A) = \mathbf{n}'(A) + \mathbf{m}'(A).$$

Ceci achève la démonstration.

2. Opérations linéaires sur l'espace $L_{\mathfrak{a}}^p$. Soit ν une mesure de Radon positive sur T, et considérons l'espace $L_{\mathfrak{a}}^p(\nu)$, $1 \leqslant p < +\infty$.[4] Soit f une application linéaire de $L_{\mathfrak{a}}^p(\nu)$ dans F'. Posons

$$|||f||| = \sup \sum_i |f(\varphi_{A_i}\mathbf{x}_i)|$$

le sup. étant considéré pour tous les champs étagés[5] $\mathbf{x} = \sum_i \varphi_{A_i}\mathbf{x}_i$ tels que les A_i soient disjoints et que $N_p(\mathbf{x}, \nu) \leqslant 1$. On a

$$\|f\| \leqslant |||f||| \leqslant +\infty.$$

Remarques.

1°. Si $p = 1$ on a toujours $\|f\| = |||f|||$. En effet, si $\|f\| = +\infty$, on a aussi $|||f||| = +\infty$. Supposons donc que $\|f\| < +\infty$. Pour tout champ étagé $\mathbf{x} = \sum_i \varphi_{A_i}\mathbf{x}_i$ tel que $N_1(\mathbf{x}, \nu) \leqslant 1$ on a

$$\sum_i |f(\varphi_{A_i}\mathbf{x}_i)| \leqslant \sum_i \|f\|\, N_1(\varphi_{A_i}\mathbf{x}_i, \nu) = \|f\|\, N_1(\mathbf{x}, \nu) \leqslant \|f\|$$

donc $|||f||| \leqslant \|f\|$, d'où il s'ensuit que $\|f\| = |||f|||$.

2°. Si $1 \leqslant p < +\infty$ et $F = C$, on a toujours $\|f\| = |||f|||$.

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[4] Pour la définition des espaces $L_{\mathfrak{a}}^p(\nu)$ voir **(14)**.

[5] On peut supposer que les $\mathbf{x}_i$ sont continus, non nécessairement de $\mathfrak{A}$.

Supposons $||f|| < +\infty$. Soit

$$\mathbf{x} = \sum_i \varphi_{A_i}\mathbf{x}_i$$

un champ étagé tel que les A_i soient disjoints et que $N_p(\mathbf{x}, \nu) \leqslant 1$. Pour chaque i, il existe un nombre complexe θ_i tel que $|\theta_i| = 1$ et

$$|f(\varphi_{A_i}\mathbf{x}_i)| = \theta_i f(\varphi_{A_i}\mathbf{x}_i) = f(\varphi_{A_i}\theta_i\mathbf{x}_i).$$

Alors

$$\sum_i |f(\varphi_{A_i}\mathbf{x}_i)| = f\Big(\sum_i \varphi_{A_i}\theta_i\mathbf{x}_i\Big) \leqslant ||f||\, N_p\Big(\sum_i \varphi_{A_i}\theta_i\mathbf{x}_i, \nu\Big) \leqslant ||f||$$

donc $|||f||| \leqslant ||f||$, et par suite $||f|| = |||f|||$.

3°. Si $1 < p < +\infty$ et $F \neq C$, on peut avoir $||f|| < |||f|||$.

Prenons $T = R$, ν la mesure de Lebesgue sur T, $E(t) = E = R$ quel que soit $t \in T$ et

$$F = L^q(\nu), \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Si f est l'application identique de $L^p(\nu)$ dans F', on a évidemment $||f|| = 1$. Montrons que $|||f||| = +\infty$. En effet, pour chaque n, on peut choisir n intervals disjoints $(A_i)_{1\leqslant i\leqslant n}$ et n nombres $(x_i)_{1\leqslant i\leqslant n}$ tels que $\nu(A_i) = 1/n$ et $|x_i| = 1$ pour $i = 1, 2, \ldots, n$. Pour la fonction étagée

$$x = \sum_{i=1}^n \varphi_{A_i}x_i$$

on a $N_p(x, \nu) = 1$ et

$$\sum_{i=1}^n |f(\varphi_{A_i}x_i)| = \sum_{i=1}^n N_p(\varphi_{A_i}x_i, \nu) = \sum_{i=1}^n [\nu(A_i)]^{1/p}\,|x_i| = n\left(\frac{1}{n}\right)^{1/p}.$$

On déduit alors que $|||f||| = +\infty$.

4°. L'ensemble des applications linéaires $f: L_\mathfrak{a}^p(\nu) \to F'$ telles que $|||f||| < +\infty$ est un espace vectoriel et $|||f|||$ est une norme sur cet espace.

THÉORÈME 6. *Une application linéaire f de $L_\mathfrak{a}^p(\nu)$ dans F' peut s'écrire sous la forme*

$$\langle z, (f\mathbf{x})\rangle = \int \langle z, U_f(t)\mathbf{x}(t)\rangle\, d\nu(t) \qquad \text{pour } \mathbf{x} \in L_\mathfrak{a}^p(\nu) \text{ et } z \in F$$

où $U_f \in \mathfrak{C}(\mathfrak{G})$ et

$$N_q(U_f, \nu) < +\infty, \quad \frac{1}{p} + \frac{1}{q} = 1,$$

si et seulement si $|||f||| < +\infty$. Dans ce cas U_f est déterminé localement ν-presque partout et $|||f||| = N_q(U_f, \nu)$.

Démonstration. Supposons d'abord qu'il existe un champ d'opérations

$U_f \in \mathfrak{C}(\mathfrak{G})$ tel que $N_q(U_f, \nu) < +\infty$ et que pour tout $z \in F$ et $\mathbf{x} \in L_q^p(\nu)$ la fonction $t \to \langle z, U_f(t)\mathbf{x}(t)\rangle$ soit essentiellement ν-intégrable et

$$\langle z, f(\mathbf{x})\rangle = \int \langle z, U_f(t)\mathbf{x}(t)\rangle \, d\nu(t).$$

Montrons alors que $|||f||| < +\infty$. Remarquons d'abord que la fonction $t \to ||U_f(t)||$ est ν-mesurable (remarque 1° suivant le théorème 2) donc appartient à $L^q(\nu)$, et

$$|\langle z, f(\mathbf{x})\rangle| \leqslant \int |\langle z, U_f(t)\mathbf{x}(t)| \, d\nu(t)$$

$$\leqslant \int |z| \, ||U_f(t)|| \, |\mathbf{x}(t)| \, d\nu(t) = |z| \int ||U_f(t)|| \, |\mathbf{x}(t)| \, d\nu(t)$$

donc

$$|f(\mathbf{x})| \leqslant \int ||U_f(t)|| \, |\mathbf{x}(t)| \, d\nu(t) \qquad \text{pour } \mathbf{x} \in L_q^p(\nu).$$

Soit

$$\mathbf{x} = \sum_i \varphi_{A_i} \mathbf{x}_i$$

un champ de vecteurs étagé tel que les A_i soient disjoints. Alors

$$\sum_i |f(\varphi_{A_i}\mathbf{x}_i)| \leqslant \sum_i \int ||U_f(t)|| \, |\mathbf{x}_i(t)| \, \varphi_{A_i}(t) \, d\nu(t)$$

$$= \int ||U_f(t)|| \left| \sum_i \varphi_{A_i}(t)\mathbf{x}_i(t) \right| d\nu(t) \leqslant N_q(U_f, \nu) \, N_p(\mathbf{x}, \nu)$$

donc

$$|||f||| \leqslant N_q(U_f, \nu) < +\infty.$$

Inversement, supposons que $|||f||| < +\infty$. Pour tout $A \in \mathfrak{B}$ et[6] $\mathbf{x} \in \mathfrak{a}$ on a $\varphi_A \mathbf{x} \in L_q^p(\nu)$. Posons

$$\mathbf{m}(A)\mathbf{x} = f(\varphi_A \mathbf{x}).$$

Si $A \in \mathfrak{B}$ et $\mathbf{x}, \mathbf{y} \in \mathfrak{a}$ sont tels que $\varphi_A \mathbf{x} = \varphi_A \mathbf{y}$, on a

$$\mathbf{m}(A)\mathbf{x} = \mathbf{m}(A)\mathbf{y}.$$

Pour chaque $A \in \mathfrak{B}$, $\mathbf{m}(A)$ est une application linéaire de $\mathfrak{a}$ dans F'. Elle est continue, car si $\mathbf{x} \in \mathfrak{a}$, $A \in \mathfrak{B}$ et $||\mathbf{x}||_A \leqslant 1$, alors

$$|\mathbf{m}(A)\mathbf{x}| = |f(\varphi_A \mathbf{x})| \leqslant ||f|| \left(\int \varphi_A(t) \, |\mathbf{x}(t)|^p \, d\nu(t) \right)^{1/p} \leqslant ||f|| \, N_p(\varphi_A, \nu)$$

donc $\mathbf{m}(A) \in \mathfrak{L}(\mathfrak{a}, F')$. On déduit que $\mathbf{m}$ est une fonction vectorielle d'ensemble et

$$||\mathbf{m}(A)|| \leqslant ||f|| N_p(\varphi_A, \nu).$$

[6] Si $\mathfrak{A}$ ne vérifie pas l'axiome (i), on peut lui ajouter de champs continus pour obtenir une famille équivalente $\mathfrak{A}'$ vérifiant (i).

De cette inégalité il résulte que $\mathbf{m}$ est dénombrablement additive, donc une mesure vectorielle. De la condition $|||f||| < +\infty$ on déduira que $\mathbf{m}$ est à variation finie. En effet, soit $A \in \mathcal{B}$, $(A_i)_{1\leq i\leq n}$ une famille d'ensembles disjoints de $\mathcal{B}$ dont la réunion est A et $(c_i)_{1\leq i\leq n}$ une famille de nombres complexes. Soit encore $\epsilon > 0$. Pour chaque i, $1 \leqslant i \leqslant n$, il existe $\mathbf{x}_i \in \mathfrak{a}$ tel que $\|\mathbf{x}_i\|_A \leqslant 1$ et tel en outre que

$$\|\mathbf{m}(A_i)\| \leqslant |\mathbf{m}(A_i)\mathbf{x}_i| + \frac{\epsilon}{|c_i|n} \qquad \text{si } c_i \neq 0$$

$$\|\mathbf{m}(A_i)\| \leqslant |\mathbf{m}(A_i)\mathbf{x}_i| + \frac{\epsilon}{n} \qquad \text{si } c_i = 0.$$

Alors, pour tout i on a

$$\|\mathbf{m}(A_i)\|\,|c_i| \leqslant |\mathbf{m}(A_i)\mathbf{x}_i c_i| + \frac{\epsilon}{n}$$

donc

$$\sum_{i=1}^{n} \|\mathbf{m}(A_i)\|\,|c_i| \leqslant \sum_{i=1}^{n} |\mathbf{m}(A_i)\mathbf{x}_i c_i| + \epsilon$$

$$= \sum_{i=1}^{n} |f(\varphi_{A_i} c_i \mathbf{x}_i)| + \epsilon \equiv |||f|||\, N_p\Big(\sum_{i=1}^{n} \varphi_{A_i} c_i \mathbf{x}_i, \nu\Big) + \epsilon$$

$$\leqslant |||f|||\, N_p\Big(\sum_{i=1}^{n} \varphi_{A_i} c_i, \nu\Big) + \epsilon.$$

Comme ϵ est arbitraire, on a

$$\sum_{i=1}^{n} \|\mathbf{m}(A_i)\|\,|c_i| \leqslant |||f|||\, N_p\Big(\sum_{i=1}^{n} \varphi_{A_i} c_i, \nu\Big).$$

En particulier, pour $c_i = 1$, $i = 1, 2, \ldots, n$, on obtient

$$\sum_{i=1}^{n} \|\mathbf{m}(A_i)\| \leqslant |||f|||\, N_p(\varphi_{A_i}, \nu)$$

donc

$$\mu(A) \leqslant |||f|||\mathrm{N}_p(\varphi_A, \nu) < +\infty$$

c'est-à-dire, $\mathbf{m}$ est à variation finie μ; en outre μ est absolument continue par rapport à ν, donc μ est régulière. Alors $\mathbf{m}$ est aussi absolument continue par rapport à ν et régulière.

On a aussi l'inégalité

$$\sum_{i=1}^{n} \mu(A_i)\,|c_i| \leqslant |||f|||\, N_p\Big(\sum_{i=1}^{n} \varphi_{A_i} c_i, \nu\Big)$$

qui sera utilisée dans la suite. En effet, soit $\epsilon > 0$. Pour chaque i, il existe une famille finie (B_{ij}) d'ensembles disjoints de $\mathcal{B}$ dont la réunion est A_i, telle que

$$\mu(A_i) \leqslant \sum_j ||\mathbf{m}(B_{ij})|| + \frac{\epsilon}{|c_i| n} \qquad \text{si } c_i \neq 0$$

$$\mu(A_i) \leqslant \sum_j ||\mathbf{m}(B_{ij})|| + \frac{\epsilon}{n} \qquad \text{si } c_i = 0.$$

Alors, pour chaque i on a

$$\mu(A_i)\,|c_i| \leqslant \sum_j ||\mathbf{m}(B_{ij})|| + \frac{\epsilon}{n}$$

donc

$$\sum_{i=1}^{n} \mu(A_i)\,|c_i| \leqslant \sum_{i,j} ||\mathbf{m}(B_{ij})||\,|c_i| + \epsilon$$

$$\leqslant |||f|||\, N_p\Big(\sum_{i,j} \varphi_{B_{ij}} c_i, \nu\Big) + \epsilon = |||f|||\, N_p\Big(\sum_{i=1}^{n} \varphi_{A_i} c_i, \nu\Big).$$

ϵ étant arbitraire, on déduit alors

$$\sum_{i=1}^{n} \mu(A_i)\,|c_i| \leqslant |||f|||\, N_p\Big(\sum_{i=1}^{n} \varphi_{A_i} c_i, \nu\Big).$$

Puisque $\mathbf{m}$ est à variation finie et absolument continue par rapport à ν, il existe un champ d'opérations $V_{\mathbf{m}} \in \mathfrak{L}(\mathfrak{H})$ tel que

$$\langle z, \int \mathbf{x}\, d\mathbf{m}\rangle = \int \langle z, V_{\mathbf{m}}(t)\mathbf{x}(t)\rangle\, d\nu(t)$$

pour $z \in F$ et $\mathbf{x} \in L_{\mathfrak{a}}^1(\mathbf{m})$, et

$$\int \varphi\, d\mu = \int ||V_{\mathbf{m}}(t)||\, \varphi(t)\, d\nu(t) \qquad \text{pour } \varphi \in L^1(\mu).$$

On prend $U_f = V_{\mathbf{m}}$, donc

$$\int \varphi\, d\mu = \int ||U_f(t)||\, \varphi(t)\, d\nu(t) \qquad \text{pour } \varphi \in L^1(\mu).$$

Soit

$$\varphi = \sum_i \varphi_{A_i} c_i$$

une fonction réelle étagée définie sur T, telle que les A_i soient disjoints; φ est μ-intégrable, donc

$$\left|\int ||U_f(t)||\, \varphi(t)\, d\nu(t)\right| = \left|\int \varphi(t)\, d\mu(t)\right| \leqslant \sum \mu(A_i)\,|c_i| \leqslant |||f|||\, N_p(\varphi, \nu).$$

On déduit alors que la fonction $t \to ||U_f(t)||$ est de puissance $q$ème intégrable pour ν et que

$$N_q(U_f, \nu) \leqslant |||f||| < +\infty.$$

Il s'ensuit que $L_{\mathfrak{a}}^p(\nu) \subset L_{\mathfrak{a}}^1(\mu)$, donc

$$\left\langle z, \int \mathbf{x}\, d\mathbf{m}\right\rangle = \int \langle z, U_f(t)\mathbf{x}(t)\rangle\, d\nu(t) \qquad \text{pour } \mathbf{x} \in L_{\mathfrak{a}}^p(\nu) \text{ et } z \in F.$$

Si

$$\mathbf{x} = \sum_i \varphi_{A_i}\mathbf{x}_i$$

est un champ de vecteurs étagé tel que les A_i soient disjoints et que $N_p(\mathbf{x},\nu)<1$, on a,

$$\sum_i |f(\varphi_{A_i}\mathbf{x}_i)| = \sum_i |\mathbf{m}(A_i)\mathbf{x}_i| = \sum_i \left|\int \varphi_{A_i}\mathbf{x}_i d\mathbf{m}\right|$$

$$\leqslant \sum_i \int ||U_f(t)||\, \varphi_{A_i}(t)\, |\mathbf{x}_i(t)|\, d\nu(t)$$

$$= \int ||U_f(t)|| \left| \sum_i \varphi_{A_i}(t)\mathbf{x}_i(t) \right| d\nu(t) \leqslant N_q(U_f,\nu) N_p(\mathbf{x},\nu) \leqslant N_q(U_f,\nu)$$

donc $|||f||| \leqslant N_q(U_f,\nu)$ et donc

$$|||f||| = N_q(U_f,\nu).$$

Pour tout champ étagé

$$\mathbf{x} = \sum_i \varphi_{A_i}\mathbf{x}_i$$

on a

$$\int \mathbf{x}\, d\mathbf{m} = \sum_i \mathbf{m}(A_i)\mathbf{x}_i = \sum_i f(\varphi_{A_i}\mathbf{x}_i) = f\left(\sum_i \varphi_{A_i}\mathbf{x}_i\right) = f(\mathbf{x}).$$

Si $\mathbf{x} \in L_q^p(\nu)$, il existe une suite $(\mathbf{x}_n)$ de champ de vecteurs étagés telle que $N_p(\mathbf{x}-\mathbf{x}_n,\nu)\to 0$. Alors $f(\mathbf{x}_n)\to f(\mathbf{x})$. D'autre part, $\mathbf{x}\in L_q^1(\mu)$ et

$$N_1(\mathbf{x}-\mathbf{x}_n,\mu) = \int |\mathbf{x}(t)-\mathbf{x}_n(t)|\, d\mu(t) \leqslant |||f|||\, N_p(\mathbf{x}-\mathbf{x}_n,\nu)$$

donc $N_1(\mathbf{x}-\mathbf{x}_n,\mu)\to 0$. Alors $\int \mathbf{x}_n d\mathbf{m}\to\int \mathbf{x} d\mathbf{m}$ et comme $f(\mathbf{x}_n)=\int \mathbf{x}_n d\mathbf{m}$, on déduit $f(\mathbf{x})=\int \mathbf{x} d\mathbf{m}$, donc

$$\langle z, f(\mathbf{x})\rangle = \int \langle z, U_f(t)\mathbf{x}(t)\rangle\, d\nu(t) \qquad \text{pour } \mathbf{x}\in L_q^p(\nu) \text{ et } z\in F.$$

L'unicité de U_f se déduit comme dans la démonstration du théorème 2. Ceci achève la démonstration.

Remarques.

1°. Si F' est de type dénombrable, on a

$$f(\mathbf{x}) = \int U_f(t)\mathbf{x}(t)\, d\nu(t) \qquad \text{pour } \mathbf{x}\in L_q^p(\nu).$$

2°. S'il existe une famille fondamentale $\mathfrak{D}\subset \mathfrak{C}(\mathfrak{G})$ vérifiant l'axiome (G) et la condition (T), alors $U_f\in L_{\mathfrak{D}}^q(\nu)$.

3°. Si l'espace F ou si les espaces $E(t)$ ne sont pas de type dénombrable, il est possible que U_f ne soit pas uniquement déterminé localement ν-presque partout.

4°. Si f est une application linéaire de $L_a^p(\nu)$ dans un espace de Banach qui n'est pas le dual d'un espace normé, il est possible que f n'admet pas une représentation intégrale quoique $|||f||| < +\infty$. Par exemple, pour l'application identique f de $L^1((0, +\infty), \nu)$ dans lui-même (ν désignant la mesure de Lebesgue) on a $|||f||| = ||f|| = 1$, mais f n'admet pas une représentation intégrale (voir **(21)**).

COROLLAIRE 1. *Pour toute application linéaire et continue f de $L_a^1(\nu)$ dans F', il existe un champ d'opérations $U_f \in \mathfrak{C}(\mathfrak{G})$ tel que $||f|| = N_\infty(U_f, \nu)$ et que*

$$\langle z, f(\mathbf{x})\rangle = \int \langle z, U_f(t)\mathbf{x}(t)\rangle \, d\nu(t) \qquad \text{pour } \mathbf{x} \in L_a^1(\nu) \text{ et } z \in F.$$

En effet, pour $p = 1$ on a $|||f||| = ||f|| < +\infty$.

Ce résultat a été démontré dans **(6)** pour le cas où F' est de type dénombrable.

COROLLAIRE 2. *Pour toute fonctionnelle linéaire et continue f sur $L_a^p(\nu)$, il existe un champ de fonctionnelles $\mathbf{x}_f' \in \mathfrak{C}(\mathfrak{E}')$ tel que $N_q(\mathbf{x}_f', \nu) = ||f||$ et que pour tout $\mathbf{x} \in L_a^p(\nu)$ on ait*

$$f(\mathbf{x}) = \int \langle \mathbf{x}(t), \mathbf{x}_f'(t)\rangle \, d\nu(t).$$

En effet, dans ce cas $|||f||| = ||f|| < +\infty$, et $F = C$. Ce résultat a été démontré par C. Ionescu Tulcea **(15)**.

BIBLIOGRAPHIE

1. S. Bochner and A. E. Taylor, *Linear functionals on certain spaces of abstractly-valued functions*, Annals of Math., *39* (1938), 913–944.
2. N. Bourbaki, *Intégration*, Chap. I–IV (1952), Chap. V (1956).
3. J. Dieudonné, *Sur le théorème de Lebesgue Nikodym (III)*, Annales Univ. Grenoble, *23* (1947–48), 25–53.
4. ——— *Sur le théorème de Lebesgue Nikodym, IV*, J. Indian Math. Soc., *15* (1951), 77–86.
5. ——— *Sur le théorème de Lebesgue Nikodym, V*, Can. J. Math., *3* (1951), 129–140.
6. N. Dinculeanu, *Sur la représentation intégrale de certaines opérations linéaires*, C. R. Acad. Sci. Paris, *245* (1957), 1203–1255.
7. ——— *Sur la représentation intégrale de certaines opérations linéaires, II*, Compositio Mathematica, *14* (1959), 1–22.
8. ——— *Sur la repésentation intégrale de certaines opérations linéaires, III*, Proc. Amer. Math. Soc., *10* (1959), 59–68.
9. ——— *Mesures vectorielles et opérations linéaires*, C. R. Acad. Sci. Paris, *246* (1958), 2328–2331.
10. ——— *Mesures vectorielles sur les espaces localement compacts*, Bull. Math. de la Soc. Sci. Math. Phys. de la R.P.R., *2* (1958), 137–164.
11. N. Dinculescu et C. Foias, *Mesures vectorielles et opérations linéaires sur L_B^p*, C. R. Acad. Sci., Paris, *248* (1959), 1759–1762.
12. N. Dunford and B. J. Petis, *Linear operations on summable functions*, Trans. Amer. Math. Soc., *47* (1940), 323–392.

13. C. Foias, *Décompositions intégrales des familles spectrales et semi-spectrales en opérateurs qui sortent de l'espace hilbertien*, Acta Sc. Math. Szeged, *20* (1959), 117–154.
14. R. Godement, *Sur la théorie des représentations unitaires*, Annals of Math., *53* (1951), 68–124.
15. C. T. Ionescu Tulcea, *Deux théorèmes concernant certains espaces de champs de vecteurs*, Bull. Sci. Math., *79* (1955), 106–111.
16. R. S. Phillips, *On linear transformation*, Trans. Amer. Math. Soc., *48* (1940), 516–541.
17. R. G. Bartle, N. Dunford, and S. Schwarz, *Weak compactness and vector measures*, Can. J. Math., *7* (1955), 289–305.
18. M. Gowurin, *Ueber die Stieltjesche integration abstrakter Funktionen*, Fundamenta Math., *27* (1936), 254–268.
19. A. Grothendieck, *Sur les applications linéaires faiblement compactes d'un espace de type C(K)*, Can. J. Math., *5* (1953), 129–173.
20. S. Kakutani, *Concrete representation of abstract (M)-spaces*, Annals of Math., *42* (1941), 994–1024.
21. S. Zaidman, *Sur la représentation des fonctions vectorielles par des intégrales de Laplace*, Bull. Math. Soc. Sci. Math. Phys. de la R.P.R., Tome 2 (*50*), (1958), 479–483.
22. A. I. Tulcea and C. I. Tulcea, *On the decomposition and integral representation af continuous linear operators*, Annali di Matematica pura ed applicata, *53* (1961), 63–87 (Ajouté sur les epreuves).

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ON NILPOTENT PRODUCTS OF CYCLIC GROUPS. II

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Introduction. In a previous paper **(18)**,[1] $G = F/F_n$ was studied for F a free product of a finite number of cyclic groups, and F_n the normal subgroup generated by commutators of weight n. In that paper the following cases were completely treated:

(a) F a free product of cyclic groups of order p^{α_i}, p a prime, α_i positive integers, and $n = 4, 5, \ldots, p + 1$.

(b) F a free product of cyclic groups of order 2^{α_i}, and $n = 4$.

In this paper, the following case is completely treated:

(c) F a free product of cyclic groups of order p^{α_i}, p a prime, α_i positive integers, and $n = p + 2$.

(Note that $n = 2$ is well known, and $n = 3$ was studied by Golovin **(2)**.) By "completely treated" is meant: a unique representation of elements of the group is given, and the order of the group is indicated. In the case of $n = 4$, a multiplication table was given.

In view of the well-known decomposition of finite nilpotent groups into direct products of p-groups, cases (a), (b), and (c) can be summarized:

(d) F a free product of cyclic groups of order α_i where the prime factors of $\alpha_i \geqslant n - 2$.

Since the results, equations, and bibliography of **(18)** are used extensively in this paper, the numbering of theorems, equations, and bibliography will be a continuation of that of **(18)**. For example, (18) and (29) refer to equations (18) and (29) of **(18)**, and this paper starts with equation (31). The same notation will also be used.

Section 1 gives preliminary results. Lemma 4 may be of particular interest, as it is an application of P. Hall's collecting process, and may be of use in attacking other group-theoretic problems. Section 2 is an exposition of the "idea" of this paper; the device used in **(18)** to deal with case (b) can be applied to (c). In § 3, case (c) is handled.

1.

LEMMA 4. *Let a, b be any two elements of a nilpotent group, and r, s any two positive integers. Then*

$$b^s a^r = a^r b^s (b,a)^{rs} ((b,a),a)^{\binom{r}{2}s} ((b,a),b)^{r\binom{s}{2}} \ldots u_i^{f_i} \ldots \tag{31}$$

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[1]Reference numbers refer to, or are continued from, this paper.

where if u_i is a standard commutator involving α a's and β b's, then

$$f_i = \sum_{\substack{j<\alpha \\ k<\beta}} n_{ijk}\binom{r}{j}\binom{s}{k},$$

n_{ijk} non-negative integers. In particular, if

$$u_i = (((b, \underbrace{a), a), \ldots, a}_{\alpha}),$$

then $f_i = \binom{r}{\alpha} s$, and if

$$u_j = ((((b, a), \underbrace{b), b), \ldots, b}_{\beta-1}),$$

then $f_j = r\binom{s}{\beta}$.

Proof. The proof is exactly the same as that given on pp. 179–181 of **(16)**. However, instead of (12.3.1) on p. 179, we have

$$b(1)b(2)\ldots b(s)a(1)a(2)\ldots a(r) \tag{32}$$

that is, the s b's and the r a's are each labelled. The precedence conditions at this first stage become

$$\begin{aligned} &b(\lambda) \text{ precedes } b(\mu) \text{ if } \lambda < \mu \\ &b(\lambda) \text{ precedes } a(\mu) \text{ if } \lambda < \mu \text{ or } \lambda = \mu \text{ or } \lambda > \mu \\ &a(\lambda) \text{ precedes } a(\mu) \text{ if } \lambda < \mu. \end{aligned} \tag{33}$$

After the a's have been collected (32) becomes an expression of the form

$$a^r b(b, a)(b, a)((b, a), a)\ldots b(b, a)(b, a)((b, a), a)\ldots \tag{34}$$

where each b and each a have the same label as before; each (b, a) has a label of the form (λ, μ); $1 \leqslant \lambda \leqslant s$; $1 \leqslant \mu \leqslant r$; it arose when $a(\mu)$ was collected: $b(\lambda)a(\mu) = a(\mu)b(\lambda)(b, a)(\lambda, \mu)$. In general, if u_j is being collected and $u_i > u_j$, and if u_i has the label $(\lambda_1, \ldots, \lambda_\sigma)$ and u_j has the label $(\mu_1, \ldots, \mu_\omega)$, then (u_i, u_j) will have the label $(\lambda_1, \ldots, \lambda_\sigma, \mu_1, \ldots, \mu_\omega)$. The existence and precedence conditions at each stage of the collecting process are all conditions (L) as described on p. 180 of **(16)**, and the induction proof given there goes through in exactly the same way to give Lemma 4. That

$$(((b, \underbrace{a), \ldots, a}_{\alpha})$$

will have the exponent

$$s\binom{r}{\alpha}$$

follows from the fact that all $(\lambda_1, \mu_1, \ldots, \mu_\alpha)$ which can be associated with such $(((b, a), \ldots, a)$ satisfy the conditions: $1 \leqslant \lambda_1 \leqslant s$; $1 \leqslant \mu_1 < \mu_2 < \ldots < \mu_\alpha \leqslant r$. A similar argument holds for $((((b, a), b)\ldots b)$.

LEMMA 5.

$$\left(\binom{r}{i}\binom{s}{j}\right)_k = \sum_{\substack{\alpha \leq ik \\ \beta \leq jk}} n_{\alpha\beta}\binom{r}{\alpha}\binom{s}{\beta},$$

$n_{\alpha\beta}$ non-negative integers.

Proof. The following set, S, has order $\binom{r}{i}\binom{s}{j}$:

$$S = \left\{(a_1, \ldots, a_i, b_1, \ldots, b_j)\ a_t, b_u \text{ integers } \begin{matrix} 1 \leqslant a_1 < \ldots < a_i \leqslant r \\ 1 \leqslant b_1 < \ldots < b_j \leqslant s \end{matrix}\right\}.$$

Let T consist of all subsets of order k taken from S;

$$T = \{(a_1^{(1)}, \ldots, a_i^{(1)}, b_1^{(1)}, \ldots, b_j^{(1)}), \ldots, (a_1^{(k)}, \ldots, a_i^{(k)}, b_1^{(k)}, \ldots, b_j^{(k)})\}.$$

T can be partitioned into disjoint subsets, depending on which of the $a_t^{(s)}$ or $b_u^{(s)}$ are equal to, greater than, or less than each other. Each such subset has order

$$\binom{r}{\alpha}\binom{s}{\beta}, \alpha \leqslant ik, \beta \leqslant jk.$$

This is sufficient to prove the lemma.

LEMMA 6. *Let A be any rational integer, p any prime and α any positive integer. Then*

$$\binom{A + p^\alpha}{p - 1} \equiv \binom{A}{p - 1} \pmod{p^\alpha}. \tag{35}$$

Proof. By definition

$$\binom{A + p^\alpha}{p - 1} = \frac{(A + p^\alpha)(A - 1 + p^\alpha) \ldots (A - p + 1 + p^\alpha)}{(p - 1)!}$$

$$= \binom{A}{p - 1} + \frac{p^\alpha u}{(p - 1)!}$$

where u is an integer. Since both

$$\binom{A + p^\alpha}{p - 1} \text{ and } \binom{A}{p - 1}$$

are integers, $p^\alpha u/(p - 1)!$ is an integer. Since $(p - 1)!$ and p^α are relatively prime, $(p - 1)!$ divides u. This is sufficient to prove Lemma 6.

LEMMA 7. *Let A, p, α be as in Lemma 6. Then*

$$\binom{A + p^\alpha}{p} \equiv \binom{A}{p} + p^{\alpha - 1} \pmod{p^\alpha} \tag{36}$$

Proof. Lemma 7 is true for $A = 0$, since $\binom{0}{p} = 0$, and

$$\binom{p^\alpha}{p} = p^{\alpha-1}\binom{p^\alpha - 1}{p-1} = p^{\alpha-1}(1 + sp) \equiv p^{\alpha-1} \pmod{p^\alpha}. \tag{37}$$

$$\binom{p^\alpha - 1}{p-1} = 1 + sp, \tag{38}$$

where s is some integer, is justified with the use of Wilson's theorem (see p. 259 of **(17)**), that is, $(p-1)! \equiv -1 \pmod{p}$. The numerator and denominator of the left-hand side of (38) consist of $p-1$ consecutive integers relatively prime to p, and the use of Wilson's theorem along with the fact that the integers modulo p form a field completes the proof of (37) and (38). This proves Lemma 7 for $A = 0$. Suppose true for A, then using induction and Lemma 6

$$\binom{A+1+p^\alpha}{p} = \binom{A+p^\alpha}{p} + \binom{A+p^\alpha}{p-1} \equiv \binom{A}{p} + p^{\alpha-1} + \binom{A}{p-1} \pmod{p^\alpha} \tag{39}$$

$$\binom{A+1}{p} = \binom{A}{p} + \binom{A}{p-1}. \tag{40}$$

Combining (39) and (40) gives the proof of Lemma 7 for A a positive integer. A similar argument proves Lemma 7 for A a negative integer.

LEMMA 8. *Let A, B, C, D be rational integers, p a prime, and α a positive integer. If*

$$A \equiv C \pmod{p^\alpha}$$

and

$$B \equiv D \pmod{p^\alpha},$$

then

$$AB - p\binom{A}{p}B - p\binom{B}{p}A \equiv CD - p\binom{C}{p}D - p\binom{D}{p}C \pmod{p^{\alpha+1}}. \tag{41}$$

Proof. It is sufficient to prove (41) for the cass $C = A + p^\alpha$ and $B = D$ in view of the symmetry of (41). Then (41) becomes

$$AB - p\binom{A}{p}B - p\binom{B}{p}A \equiv (A+p^\alpha)B - p\binom{A+p^\alpha}{p}B - p\binom{B}{p}(A+p^\alpha) \pmod{p^{\alpha+1}}. \tag{42}$$

If one expands the right-hand side of (42) and makes use of Lemma 7, one obtains the left-hand side of (42) modulo $p^{\alpha+1}$. Note that since

$$\binom{A+p^\alpha}{p} \equiv \binom{A}{p} + p^{\alpha-1} \pmod{p^\alpha},$$

$$p\binom{A+p^\alpha}{p} \equiv p\binom{A}{p} + p^\alpha \pmod{p^{\alpha+1}}.$$

2. In this section, the "idea" of the proof will be explained. Details will be carried out in § 3.

In § 2 of **(18)**, the "well-behaved" case was studied. In F, a free group or a free product of cyclic groups, a sequence of standard commutators $u_1, \ldots$ was selected and it was shown that every element of F/F_n could be written uniquely as $\prod u_k^{c_k}$. In the case of F/F_4 (18) gives the multiplication table for two such elements. One reason for the failure of the proof for the case $p = 2$ with F/F_4 is that in (18), terms such as $\binom{d_i}{2}$ appear which are not unique modulo the appropriate powers of 2. To get around this difficulty, another set of basis elements was chosen in § 3 to handle the case of $p = 2$. When the new basis is used (29) is the multiplication table of two elements of F/F_4. Actually, (29) can be considered a modification of (18) in the following sense: Let $F = \{a, b\}$ be a free group on two generators. Then every element g of F/F_4 can be expressed (uniquely) as

$$a^{e_1}b^{e_2}(a,b)^{e_{12}}((a,b),a)^{e_{121}}((a,b),b)^{e_{122}} \tag{43}$$

where e_i, e_{12}, e_{ijk} are rational integers. When (28) is used every element of F/F_4 can be expressed uniquely as

$$a^{\gamma_1}b^{\gamma_2}(a,b)^{\gamma_{12}}(a^2,b)^{\gamma_{121}}(a,b^2)^{\gamma_{122}} \tag{44}$$

where γ_i, γ_{12}, γ_{ijk} are rational integers. Since

$$\begin{aligned} &a^{e_1}b^{e_2}(a,b)^{e_{12}}((a,b),a)^{e_{121}}((a,b),b)^{e_{122}} \\ &\quad = a^{e_1}b^{e_2}(a,b)^{e_{12}}((a^2,b)^{e_{121}}(a,b)^{-2e_{121}}(a,b^2)^{e_{122}}(a,b)^{-2e_{122}} \\ &\quad = a^{e_1}b^{e_2}(a,b)^{e_{12}-2e_{121}-2e_{122}}(a^2,b)^{e_{121}}(a,b^2)^{e_{122}}, \end{aligned}$$

$$\begin{array}{lll} \gamma_i = e_i & & e_i = \gamma_i \\ \gamma_{12} = e_{12} - 2e_{121} - 2e_{122} & & e_{12} = \gamma_{12} + 2\gamma_{121} + 2\gamma_{122} = \alpha(\gamma_{12}) \\ \gamma_{121} = e_{121} & & e_{121} = \gamma_{121} \\ \gamma_{122} = e_{122} & & e_{122} = \gamma_{122}. \end{array} \tag{45}$$

Let

$$\begin{aligned} h &= a^{d_1}b^{d_2}(a,b)^{d_{12}}((a,b),a)^{d_{121}}((a,b),b)^{d_{122}} \\ &= a^{\delta_1}b^{\delta_2}(a,b)^{\delta_{12}}(a^2,b)^{\delta_{121}}(a,b^2)^{\delta_{122}} \end{aligned}$$

and

$$\begin{aligned} g\cdot h &= a^{e_1}b^{e_2}(a,b)^{e_{12}}((a,b),a)^{e_{121}}((a,b),b)^{e_{122}} \\ &= a^{\epsilon_1}b^{\epsilon_2}(a,b)^{\epsilon_{12}}(a^2,b)^{\epsilon_{121}}(a,b^2)^{\epsilon_{122}}. \end{aligned}$$

One can now take ϵ_{12} (or ϵ_{121} or ϵ_{122}), express it in terms of e_{12}, e_{121}, e_{122} using (45), then in terms of $c_{12}, d_{12}, c_{121}, d_{121}, c_{122}, d_{122}$ using (18) with $i = 1$ and $j = 2$, and then in terms of $\gamma_{12}, \delta_{12}, \gamma_{121}, \delta_{121}, \gamma_{122}, \delta_{122}$ using (45). This gives the ϵ's in terms of the γ's and the δ's. If one now substitutes e_{12}, c_{12}, d_{12} for $\epsilon_{12}, \gamma_{12}, \delta_{12}$ and $e_{12}^{(2)}, c_{12}^{(2)}, d_{12}^{(2)}$ for $\epsilon_{121}, \gamma_{121}, \delta_{121}$ and $e_{12}^{(3)}, c_{12}^{(3)}, d_{12}^{(3)}$ for

ϵ_{122}, γ_{122}, δ_{122} respectively, one obtains (29) for $i = 1$, $j = 2$. Hence (29) is a modification of (18) in the sense that one set of basic commutators has been substituted for another, but the same multiplication table has been used. Hence it is not necessary to check the group axioms for the group H, as indicated after (29); all that needs to be done is to ascertain whether (29) is unambiguous modulo appropriate powers of 2. The author did not realize this until after writing paper **(18)**.

This same idea will be used to study F/F_{p+2} where F is a free product of cyclic groups of order p^{e_i}, p a fixed prime. The over-all strategy is:

I. Investigations of the terms appearing in the multiplication table analogous to (18) for F/F_{p+2}. In particular, it will be shown that p appears in the denominator of a term only when the corresponding u_i is

$$(((b, a), \underbrace{\ldots, a}_{p})$$

or

$$((((b, a), \underbrace{b), \ldots, b}_{p-1}).$$

Call these terms u_p' and u_p'' respectively.

II. u_p' and u_p'' will be replaced by (b, a^p) and (b^p, a) respectively, and the multiplication table of F/F_{p+2} analogous to (29) will be investigated to ascertain whether the terms are unambiguous modulo appropriate powers of p.

It will be assumed that F is a free group until near the end. The proof of the unambiguity of the multiplication table modulo appropriate powers of p will be sufficient to prove the desired theorem for F a free product of cyclic p-groups.

3.

I. *Investigation of the multiplication table of $G = F/F_{p+2}$ where each element of G is expressed as a product of standard commutators.*

Let $F\{a, b\}$ be a free group with two generators, and let $g \in F/F_n$. Then g can be uniquely expressed as

$$g = a^{e_1}b^{e_2}(b, a)^{e_3}((b, a), a)^{e_4}((b, a), b)^{e_5} \ldots = \prod u_i^{c_i}$$

where u_i are a sequence of standard commutators (see **(7)**) and c_i are rational integers. Let $h \in F/F_n$ and

$$h = a^{d_1}b^{d_2}(b, a)^{d_3} \ldots = \prod u_i^{d_i}.$$

Then to compute a multiplication table of F/F_n analogous to (18), we put

$$a^{e_1}b^{e_2} \ldots = \prod u_i^{e_i} = a^{e_1}b^{e_2} \ldots a^{d_1}b^{d_2} \ldots = g \cdot h$$

and we first collect a^{d_1}, then b^{d_2} and so on. A typical step consists in

$$u_i^{e_i}u_j^{d_j} = u_j^{d_j}u_i^{e_i}(u_i, u_j)^{e_id_j}((u_i, u_j), u_j)^{e_i\binom{d_j}{2}} \ldots \tag{46}$$

where Lemma 4 (that is (31)) is used. Since the commutators appearing on the right-hand side of (46) may not be standard commutators, they should be expressed as products of standard commutators and (4) used. Using Lemma 5, and repeating until the right-hand side of (46) is a product of the original standard commutators, one obtains

$$u_i^{c_i} u_j^{d_j} = u_j^{d_j} u_i^{c_i} \prod u_k^{n_k \binom{c_i}{\alpha_k} \binom{d_j}{\beta_k}} \tag{47}$$

where n_k are non-negative integers. Note that if σ_k is the weight of u_k in a's and b's (that is, $u_k \in F_{\sigma_k}$, $\notin F_{\sigma_k+1}$), then $\alpha_k + \beta_k \leqslant \sigma_k$. The only time $\binom{c_i}{p}$ or $\binom{d_j}{p}$ appears is if $u_k = (((b, a), a, \ldots, a)$ $(p - 1$ a's) or $((((b, a), b), b), \ldots, b)$ $(p - 1, b$'s).

In general, instead of $u_i^{c_i}$ (or $u_j^{d_j}$) in (46), one may have an element of the form

$$u_i^{\binom{c_{i_1}}{\alpha_1} \ldots \binom{c_{i_s}}{\alpha_s} \binom{d_{i_1}}{\beta_1} \ldots \binom{d_{i_t}}{\beta_t}}$$

in which case the exponent of u_k on the right-hand side of (47) will be of the form

$$n_k \binom{c_{j_1}}{\alpha_1} \ldots \binom{c_{j_s}}{\alpha_s} \binom{d_{j_1}}{\beta_1} \ldots \binom{d_{j_t}}{\beta_t},$$

where n_k are non-negative integers, and

$$\sum \alpha_i + \sum \beta_j \leqslant \sigma_k$$

where u_k has weight σ_k in the a's and b's.

We now investigate under what conditions u_p' and u_p'' occur. Let $u_0' = b$, $u_1' = (b, a)$, $u_{s+1}' = (u_s', a)$ for s a positive integer. Similarly, let

$$u_1'' = (b, a), u_{s+1}'' = (u_s'', b).$$

Note that in (47), each u_k will have at least as many a's and b's as (u_i, u_j). Hence u_s' can occur in (46) and (47) only when a^{d_1} is being collected, for example

$$u_t'^{c_t'} a^{d_1} = a^{d_1} u_t'^{c_t'} \ldots u_s'^{\binom{d_1}{s-t} c_t'} \ldots t < s \tag{48}$$

(where c_t' is the exponent of u_t' in g). Here Lemma 4 is used. Hence, when u_s' is collected, its exponent will be

$$c_s' + d_s' + \sum_{t=0}^{s-1} c_t' \binom{d_1}{s-t} \tag{49}$$

(where d_s' is the exponent of u_s' in h). When a^{d_1} is collected u_s'' occurs:

$$b^{c_2} a^{d_1} = a^{d_1} b^{c_2} u_1''^{c_2 d_1} \ldots u_s''^{c_2 d_1 \binom{c_2}{s}} \ldots \tag{50}$$

and hence when b^{d_2} is collected, u_t'' appears twice, once with exponent c_t''

(its exponent from g at the beginning) and also with exponent $d_1\binom{c_2}{t}$ from (50). Hence

$$u_t''^{a}b^{d_2} = b^{d_2}u_t''^{a}\dots u_s''^{a\binom{d_2}{s-t}}.$$

After b^{d_2} is collected, u_s'' will not arise again in the subsequent collectings, so that when it is collected, its exponent will be

$$c_s'' + d_s'' + \sum_{t=1}^{s-1}\left[c_t'' + d_1\binom{c_2}{t}\right]\binom{d_2}{s-t} + d_1\binom{c_2}{s}. \tag{51}$$

The following theorem and corollaries have been proved:

THEOREM 5. *Let $F = \{a, b\}$ be the free group on two generators. Let $G = F/F_n$. Let $u_1, u_2, \dots$ be the sequence of standard monomial commutators of non-decreasing weight $< n-1$ (see (7)) in a and b. Let $g, h \in G$,*

$$g = a^{e_1}b^{e_2}(b,a)^{e_3}\dots u_i^{e_i}\dots = \prod u_i^{e_i}$$
$$h = a^{d_1}b^{d_2}(b,a)^{d_3}\dots u_i^{d_i}\dots = \prod u_i^{d_i}$$

where c_i, d_i are rational integers. If

$$g\cdot h = \prod u_i^{e_i},$$

then

$$e_t = c_t + d_t + \sum n_k\binom{c_{t_1}}{\alpha_1}\dots\binom{c_{t_s}}{\alpha_s}\binom{d_{t_1}}{\beta_1}\dots\binom{d_{t_s}}{\beta_s} \tag{52}$$

where if $u_t \in F_s$, $u_t \notin F_{s+1}$ (that is, u_t is a commutator of weight s in a and b),

$$\sum \alpha_j + \sum \beta_j \leqslant s.$$

In particular, let $(b,a) = u_1'$, $(u_s', a) = u_{s+1}'$ and c_s', d_s', e_s' be the exponents of u_s' in g, h, and $g\cdot h$ respectively, then

$$e_s' = c_s' + d_s' + \sum_{t=1}^{s-1} c_t'\binom{d_1}{s-t} + c_2\binom{d_1}{s}. \tag{49}$$

Let $(b,a) = u_1''$, $(u_s'', b) = u_{s+1}''$ with c_s'', d_s'', e_s'' the exponents of u_s'' in g, h, $g\cdot h$ respectively, then

$$e_s'' = c_s'' + d_s'' + \sum_{t=1}^{s-1}\left[c_t'' + d_1\binom{c_2}{t}\right]\binom{d_2}{s-t} + d_1\binom{c_2}{s}. \tag{51}$$

COROLLARY 1. *Let $n = p+2$, p a prime in Theorem 5. Then*

$$e_p' = c_p' + d_p' + \sum_{t=1}^{p-1} c_t'\binom{d_1}{p-t} + c_2\binom{d_1}{p}, \tag{53}$$

$$e_p'' = c_p'' + d_p'' + \sum_{t=1}^{p-1}\left[c_t'' + d_1\binom{c_1}{t}\right]\binom{d_2}{p-t} + d_1\binom{c_2}{p}, \tag{54}$$

and these are the only places where p appears in the denominator of a term of an e_t.

COROLLARY 2. *Let F be a free group on 3 or more generators, p a fixed prime, $u_1, \ldots,$ a sequence of standard commutators in the generators of F,*

$$g = \prod u_i^{c_i}, h = \prod u_i^{d_i}, g \cdot h = \prod u_i^{e_i}.$$

Then for $n = p + 2$,

$$e_i = c_i + d_i + \sum n_k \binom{c_{i_1}}{\alpha_1} \cdots \binom{c_{i_s}}{\alpha_s} \binom{d_{i_1}}{\beta_1} \cdots \binom{d_{i_t}}{\beta_t}$$

as in Theorem 5, and if u_i contains at least 3 generators of F, then $\alpha_j < p$, $\beta_j < p$.

Comment. For $u_3 = (b, a)$ $(u_1 = a, u_2 = b)$, (49) becomes

$$e_3 = c_3 + d_3 + c_2 d_1. \tag{55}$$

In (18) the corresponding formula is

$$c_3 + d_3 - c_2 d_1$$

(c_{ij}, d_{ij}, e_{ij} of (18) become c_3, d_3, e_3 here, and $j = 2$ and $i = 1$). The reason for $-c_2 d_1$ in (18) is that (a, b) is used instead of (b, a). Similar comments apply to the other equations of (18).

II. u_p' and u_p'' are replaced by (b, a^p) and (b^p, a) respectively, and the corresponding multiplication table for F/F_{p+2} is investigated.

As before, let F be a free group on two generators, $g, h \in F/F_{p+2}$. Let $u_1 = a$, $u_2 = b$, $u_3 = (b, a), \ldots,$ u_s' and u_s'' as in Theorem 5; similarly for g, h and $g \cdot h$. Let $v_i = u_i$ except that u_p' is replaced by $(b, a^p) = v_p'$, and u_p'' is replaced by $(b^p, a) = v_p''$. If one puts $s = 1$ and $r = p$ in (31), and brings $a^p b$ over to the left-hand side, one obtains

$$(b, a^p) = b^{-1} a^{-p} b a^p = (b, a)^p \prod u_i^{pf_i} u_p' \pmod{F_{p+2}} \tag{56}$$

where f_i are positive integers, that is,

$$v_p' = (b, a)^p \prod u_i^{pf_i} u_p' \pmod{F_{p+2}}. \tag{57}$$

Similarly

$$v_p'' = (b, a)^p \prod u_i^{pg_i} u_p'' \pmod{F_{p+2}} \tag{58}$$

where g_i are positive integers. Using (57) and (58), we can take any $g \in F/F_{p+2}$ and express it either in terms of u_i or v_i. If

$$g = \prod u_i^{c_i} = \prod v_i^{\gamma_i}$$

and the v_i are expressed in terms of the u_i, and terms collected, one obtains

$$\begin{aligned} c_1 &= \gamma_1 \\ c_2 &= \gamma_2 \\ c_3 &= \gamma_3 + p\gamma_p' + p\gamma_p'' \\ c_i &= \gamma_i + ph_i \qquad h_i \text{ function of the } \gamma_j \\ c_p' &= \gamma_p' \\ c_p'' &= \gamma_p'' \end{aligned} \tag{59}$$

where γ_p', γ_p'' are exponents associated with v_p' and v_p'' respectively. In order to compute the h_i, Lemma 4 and (4) must be extensively used. Equations similar to (59) can be formed for d_i and δ_i, e_i and ϵ_i where

$$h = \prod u_i^{d_i} = \prod v_i^{\delta_i} \quad \text{and} \quad g \cdot h = \prod u_i^{e_i} = \prod v_i^{\epsilon_i}.$$

We now compute ϵ_i in terms of γ_i and δ_i. Obviously

$$\epsilon_1 = \gamma_1 + \delta_1 \tag{60}$$
$$\epsilon_2 = \gamma_2 + \delta_2.$$

To find ϵ_3, we first use (59)

$$e_3 = \epsilon_3 + p\epsilon_p' + p\epsilon_p''$$

or

$$\epsilon_3 = e_3 - p\epsilon_p' - p\epsilon_p''.$$

Using (55) and (59)

$$\epsilon_3 = c_3 + d_3 + c_2 d_1 - pe_p' - pe_p'',$$

Using (53) and (54)

$$\epsilon_3 = c_3 + d_3 + c_2 d_1 - p\Big\{c_p' + d_p' + \sum_{t=1}^{p-1} c_t'\binom{d_1}{p-t} + c_2\binom{d_1}{p} + c_p'' + d_p'' + \sum_{t=1}^{p-1}\Big[c_t'' + d_1\binom{c_1}{t}\Big]\binom{d_2}{p-t} + d_1\binom{c_2}{p}\Big\}.$$

Using (59) again, and collecting terms

$$\epsilon_3 = \gamma_3 + \delta_3 + \gamma_2\delta_1 - p\gamma_2\binom{\delta_1}{p} - p\delta_1\binom{\gamma_2}{p} + p\Big\{\sum_{t=1}^{p-1}(\gamma_t' + ph_t)\binom{\delta_1}{p-t} + \sum_{t=1}^{p-1}\Big[\gamma_t'' + ph_t + \delta_1\binom{\gamma_1}{t}\Big]\binom{\delta_2}{p-t}\Big\} \tag{61}$$

A similar computation gives (using (52), (53), (54))

$$\epsilon_i = \gamma_i + \delta_i + \sum n_k\binom{\gamma_{i_1} + ph_{i_1}}{\alpha_1}\cdots\binom{\delta_{j_s} + ph_{j_s}}{\beta_s} - ph_i(\gamma_j, \delta_k) \tag{62}$$

$$\alpha_{i_t}, \beta_{j_t} < p$$

$$\epsilon_p' = \gamma_p' + \delta_p' + \sum [\gamma_t' + ph_t]\binom{\delta_1}{p-t} + \gamma_2\binom{\delta_1}{p} \tag{63}$$

$$\epsilon_p'' = \gamma_p'' + \delta_p'' + \sum\Big[\gamma_t'' + ph_t + \delta_1\binom{\gamma_1}{t}\Big]\binom{\delta_2}{p-t} + \delta_1\binom{\gamma_2}{p}. \tag{64}$$

Equations (60) through (64) are analogous to (29). We now assume that

$$a^{p^\alpha} = b^{p^\beta} = 1 \qquad \alpha < \beta,$$

and we investigate (60) through (64) with γ_i, δ_i, ϵ_i considered as integers modulo powers of p. The proof of Lemma 1 in § 1 of **(18)** shows that

$$v_i^{p^\alpha} = 1 \tag{65}$$

except if $v_i = b$, (b, a), v_p' or v_p''. Similarly

$$u_i^{p^\alpha} = 1 \tag{66}$$

except if $u_i = b$ or (b, a).

In (31), put $s = 1$, $r = p^\alpha$. This gives

$$1 = (b, a^{p^\alpha}) = (b, a)^{p^\alpha} \prod u_x^{p^\alpha} u_y'^{\binom{p^\alpha}{y}} = (b, a)^{p^\alpha} u_p'^{p^{\alpha-1}} \tag{67}$$

where Lemma 7 (that is (36)) has been used with $A = 0$. Similarly

$$1 = (b, a^{p^{\alpha+1}}) = (b, a)^{p^{\alpha+1}} u_p'^{p^\alpha} = (b, a)^{p^{\alpha+1}}. \tag{68}$$

Hence, u_3 (or v_3) has order at most $p^{\alpha+1}$. Now using (57), (4), and (67),

$$v_p'^{p^{\alpha-1}} = (b, a^p)^{p^{\alpha-1}} = [(b, a)^p \prod u_i^{p\beta_i} u_p'^{\binom{p}{2}}]^{p^{\alpha-1}} = (b, a)^{p^\alpha} u_p'^{p^{\alpha-1}} = 1.$$

If $\alpha = \beta$, then

$$v_p''^{p^{\alpha-1}} = 1$$

otherwise (that is, $\alpha < \beta$)

$$v_p''^{p^\alpha} = 1.$$

We now show that (60) through (64) are unambiguous if

$$\begin{aligned}
&\gamma_1, \delta_1, \epsilon_1 \text{ are integers modulo } p^\alpha \\
&\gamma_2, \delta_2, \epsilon_2 \text{ are integers modulo } p^\beta \\
&\gamma_3, \delta_3, \epsilon_3 \text{ are integers modulo } p^{\alpha+1} \\
&\gamma_i, \delta_i, \epsilon_i \text{ are integers modulo } p^\alpha \text{ if } i > 3, \text{ except} \\
&\gamma_p', \delta_p', \epsilon_p' \text{ are integers modulo } p^{\alpha-1} \\
&\gamma_p'', \delta_p'', \epsilon_p'' \text{ are integers modulo } \begin{cases} p^{\alpha-1} \text{ if } \alpha = \beta \\ p^\alpha \text{ if } \alpha < \beta \end{cases}.
\end{aligned} \tag{69}$$

(60) is obviously unambiguous. As for (61), consider its three parts

$$E = \gamma_3 + \delta_3$$
$$F = \gamma_2\delta_1 - p\gamma_2\binom{\delta_1}{p} - p\binom{\gamma_2}{p}\delta_1$$
$$G = \epsilon_3 - E - F$$

E gives no trouble. F is taken care of by Lemma 8. G will cause no difficulties because it has a factor of p. γ_p' or δ_p' (modulo $p^{\alpha-1}$) if they appear at all in G are in the h_i which are multiplied by p^2, and hence G is unambiguous modulo $p^{\alpha+1}$. For reasons similar to G (62) causes no trouble. The factor $\binom{\delta_1}{p}$ in

(63) is covered by Lemma 7. If $\alpha = \beta$, the factor $\binom{\gamma_2}{p}$ is covered by Lemma 7; if $\alpha < \beta$, then γ_2 is an integer modulo p^β and a similar argument applies.

We have now proved the following theorem:

THEOREM 6. *Let $A_1, A_2, \ldots, A_t$ be cyclic groups of order $p^{\alpha_1}, p^{\alpha_2}, \ldots, p^{\alpha_t}$, respectively, α_i positive integers, p a fixed prime, $\alpha_1 \leqslant \alpha_2 \leqslant \ldots \leqslant \alpha_t$. Let a_i generate A_i. Let $v_1, v_2, \ldots$ be the sequence of standard commutators of non-decreasing weight in the a_i of weight $< p+1$ (see (7)), except that*

$$(\underbrace{((a_i, a_j), \ldots, a_j)}_{p}) \quad \text{and} \quad (\underbrace{((a_i, a_j), a_i), \ldots, a_i)}_{p-1}$$

are replaced by

$$v'_{ij,p} = (a_i, a_j^p) \quad \text{and} \quad v''_{ij,p} = (a_i^p, a_j)$$

respectively. Let

$$N_i = p^{\alpha_i} \text{ if } v_i \text{ is of weight } 1,$$
$$N_i = \gcd(p^{\alpha_j}) \text{ if } a_j \text{ appears in } v_i,$$

except that for

$$v_i = v'_{ij,p}, N_i = \gcd(p^{\alpha_i - 1}, p^{\alpha_j - 1})$$

and for

$$v_i = v''_{ij,p}, N_i = \begin{cases} p^{\alpha_i - 1} & \text{if } \alpha_i = \alpha_j \\ \gcd(p^{\alpha_i}, p^{\alpha_j}) & \text{if } \alpha_i \neq \alpha_j. \end{cases}$$

Then every element g, of F/F_{p+2} can be uniquely expressed as

$$\prod v_i^{\gamma_i}$$

where γ_i are integers modulo N_i.

REFERENCES

16. Marshall Hall, *The theory of groups*, Macmillan Co. (New York, 1959).
17. Oystein Ore, *Number theory and its history*, McGraw-Hill Book Co., Inc. (New York, 1948).
18. R. R. Struik, *On nilpotent products of cyclic groups*, Can. J. Math., *12* (1960), 447–462.

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SOME PROPERTIES OF NOETHERIAN DOMAINS OF DIMENSION ONE

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Introduction. Throughout this discussion R will be an integral domain with quotient field Q and $K = Q/R \neq 0$. If A is an R-module, then A is said to be *torsion-free* (resp. *divisible*), if for every $r \neq 0 \in R$ the endomorphism of A defined by $x \to rx$, $x \in A$, is a monomorphism (resp. epimorphism). If A is torsion-free, the *rank of* A is defined to be the dimension over Q of the vector space $A \otimes_R Q$; (we note that a torsion-free R-module of rank one is the same thing as a non-zero R-submodule of Q). A will be said to be *indecomposable*, if A has no proper, non-zero, direct summands. We shall say that A has D.C.C., if A satisfies the descending chain condition for submodules. By dim R we shall mean the maximal length of a chain of prime ideals in R.

In the following remarks the ring R will be a Noetherian integral domain, unless specified otherwise.

The purpose of this paper is to study some of the properties and relationships of R-submodules of Q and R-homomorphic images of Q. With each statement concerning these objects we can associate a dual statement obtained by interchanging the words submodule and homomorphic image, torsion-free and divisible, free and injective, finitely generated and D.C.C., and the self-dual term indecomposable. We now list a series of such statements and their duals ((1′) is the dual of (1), (2′) is the dual of (2), etc.).

(1) Every proper R-submodule of Q is finitely generated.

(1′) Every proper R-homomorphic image of Q has D.C.C.

(2) No proper homomorphic image of an R-submodule of Q is torsion-free.

(2′) No proper submodule of an R-homomorphic image of Q is divisible.

(3) Every R-submodule of Q is indecomposable.

(3′) Every R-homomorphic image of Q is indecomposable.

(4) Every cyclic R-submodule of Q is free.

(4′) Every factor module of Q by a cyclic R-submodule is injective (that is, K is injective).

Statements (2), (3), and (4) are, of course, trivial for any integral domain; but their duals are highly non-trivial. One aim of this discussion is to find necessary and sufficient conditions for the validity of these statements and for their equivalence.

A most interesting feature of these statements is the close relationship they

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bear to each other and to the condition that R is a local ring of dimension one. In Theorem 1 we prove that (1′) is equivalent to R being semi-local and $\dim R = 1$. This gives us a well-known theorem of I. S. Cohen as a corollary. Condition (1) is much more restrictive, for in Theorem 2 we prove that (1) and (2′) are equivalent and imply (3′), R local, and $\dim R = 1$. However, if we replace (1) by a weaker finiteness condition on the proper R-submodules of Q, then Theorem 3 shows that we do get a condition for local rings that is equivalent to $\dim R = 1$.

We now consider the statement:

(5) $\operatorname{Ext}_R^1(Q, S) = 0$ for every R-submodule S of Q.

In Theorem 4 we show that (5) is equivalent to R being a complete, local domain of dimension one; and if R is complete and local, then (5) is equivalent to (1), (1′), (2′), and (3′). As a corollary we obtain a special case of a theorem of M. Nagata. The strength of condition (5) is not too surprising when we compare it with the fact that if R is any integral domain, then R is a maximal valuation ring if and only if $\operatorname{Ext}_R^1(A, S) = 0$ for every torsion-free R-module A and every R-submodule S of Q [Theorem A4].

Now in **(8)** we also studied the condition that every R-homomorphic image of Q is injective, which, in the case of a Noetherian domain, we showed is equivalent to R being a Dedekind ring. Condition (4′) is not equivalent to R being a Dedekind ring, but does imply $\dim R = 1$. In Thoerems 5 and 6 we study this condition and some of its implications. Finally, we show the relationship of some of the listed conditions to the existence of indecomposable, torsion-free R-modules of rank two.

To facilitate matters for the reader we shall provide in the Appendix a list of some of the theorems we shall find necessary, together with indications of where their proofs may be found. In the text a reference to Theorem A1 will mean a reference to Theorem A1 of the Appendix, etc.

Definitions and notations. Let A be an R-module. Then:

(1) $0(A) = \{r \in R \mid rA = 0\}$.

(2) If I is any ideal of R, then $\operatorname{Ann}_A(I) = \{x \in A \mid Ix = 0\}$.

(3) $E(A)$ is the injective envelope of A (see **(7)**).

(4) $\operatorname{Hd}_R A$ is the projective dimension of A as an R-module.

(5) A is said to be reduced, if A has no proper, non-zero, divisible submodules.

(6) An ideal I of a ring R is said to be irreducible, if it is not an intersection of two strictly larger ideals of R.

(7) Let L be a multiplicatively closed subset of R. Then R_L is the ring of quotients of R with respect to L, and $A_L = A \otimes_R R_L$.

If L is the complement of a prime ideal P of R, then we shall define R_P to be R_L, and A_P to be A_L.

1. K has D.C.C.

THEOREM 1. *Let R be a Noetherian integral domain. Then the following are equivalent:*

(1) *R is semi-local and* $\dim R = 1$.

(2) *K has* D.C.C.

If one of these conditions is satisfied, then every proper R-homomorphic image of Q has D.C.C.

Proof. (2) $\Rightarrow$ (1). Assume K has D.C.C. Let $P \neq 0$ be a prime ideal of R, and let $b \neq 0 \in P$. Let $x = 1/b + R, x \in K$. Then Rx has a composition series, and so there exist maximal ideals $M_1, \ldots, M_n$ of R such that if $I = M_1 M_2 \ldots M_n$, then $Ix = 0$. Thus $I \subset bR \subset P$, and so P is equal to one of the M_i's and is maximal. Thus $\dim R = 1$. Now by Theorem A7 we have $K = \sum \oplus K_{M_\alpha}$, where the M_α's range over all of the maximal ideals of R. Since K has D.C.C., there can only be a finite number of components in this direct sum, and so R is semi-local.

(1) $\Rightarrow$ (2). Assume R is semi-local and $\dim R = 1$. Let $M_1, \ldots, M_n$ be the maximal ideals of R. Then by Theorem A7 we have $K = K_{M_1} \oplus \ldots \oplus K_{M_n}$. Since a finite direct sum of modules has D.C.C. if and only if each direct summand has D.C.C., and since K_{M_i} has D.C.C. over R_{M_i} if and only if it has D.C.C. over R, we can assume that R is a local ring of dimension one.

Let M be the maximal ideal of R. Then it is easy to see that since $\dim R = 1$, $M^{-1}/R \neq 0$ and K is an essential extension of M^{-1}/R (see **(6)**). Thus $E(K) = E(M^{-1}/R)$. Now M^{-1}/R is a finite direct sum of copies of R/M, and so $E(M^{-1}/R)$ is a finite direct sum of copies of $E(R/M)$. Thus K has D.C.C. by Theorem A8.

If K has D.C.C., let $T \neq 0$ be any proper R-submodule of Q. Choose $x \neq 0 \in T$. Then we have an exact sequence:

$$Q/Rx \to Q/T \to 0.$$

Since $Q/Rx \cong K$, Q/T has D.C.C.

In order to show the connection of Theorem 1 with earlier results we derive the following theorem of I. S. Cohen **(3**, Theorem 4**)** as a corollary.

COROLLARY (*Theorem of I. S. Cohen*). *Let R be a Noetherian domain such that* $\dim R = 1$. *Let $S \neq Q$ be any ring between R and Q. Then S is Noetherian,* $\dim S = 1$, *and if R is semi-local, so is S. Furthermore, if A is any non-zero ideal of S, then S/A has a composition series as an R-module.*

Proof. Let $I = A \cap R$. Then to prove the theorem it is sufficient to prove that S/SI has a composition series as an R-module. Now by Theorem A7,

$$S/SI = \sum \oplus S_{M_\alpha}/S_{M_\alpha} I_{M_\alpha},$$

where M_α ranges over all maximal ideals of R. Since I is contained in only a finite number of maximal ideals of R, the above direct sum has only a finite number of non-zero components. Since $S_{M_\alpha}/S_{M_\alpha} I_{M_\alpha}$ has a composition series

as an R-module if and only if it has a composition series as an R_{M_a}-module, we can assume that R is a local ring with maximal ideal M. By Theorem 1, S/SI has D.C.C. over R. Since there exists an integer $n > 0$ such that $M^n \subset I$, we have $M^n(S/SI) = 0$. Thus S/SI has a composition series over R.

PROPOSITION 1. *Let R be a Noetherian, local domain such that* dim $R = 1$. *Let M be the maximal ideal of R, and D an R-module with* D.C.C. *Then D is a divisible R-module if and only if $MD = D$.*

Proof. Of course, if D is divisible, then $MD = D$. Conversely, assume that $MD = D$. Then by Theorem A9 there exists an element $s \neq 0 \in M$ such that $sD = D$. Let $b \neq 0 \in M$, and let L be the multiplicative system consisting of the powers of s. Then $R_L = Q$, and so there exists $r \in R$ such that $1/b = r/s^k$ for some integer $k > 0$. Hence we have $D = s^kD = b(rD)$, and so D is divisible.

THEOREM 2. *Let R be a Noetherian integral domain. Then the following are equivalent:*

(1) *The integral closure of R is a valuation ring, finitely generated over R.*

(2) *Every proper R-submodule of Q is finitely generated.*

(3) *No R-homomorphic image of Q has a proper, non-zero, divisible submodule.*

If any of these conditions hold, then R is local, dim $R = 1$, *and the following condition holds:*

(4) *Every R-homomorphic image of Q is indecomposable. Conversely, if condition* (4) *holds, then R is local,* dim $R = 1$, *and the integral closure of R is a discrete valuation ring.*

Proof.

(1) $\Rightarrow$ (2). Let V be the integral closure of R, and let S be any proper R-submodule of Q. Since V is finitely generated over R, there exists $r \neq 0 \in R$ such that $rV \subset R$; and so $rVS \subset S$. Thus $VS \neq Q$, and since V is a discrete valuation ring, VS is a finitely generated V-module. It follows that VS is a finitely generated R-module; and since $S \subset VS$, S is a finitely generated R-module.

(2) $\Rightarrow$ (3). This is an immediate consequence of the fact that a divisible module cannot be finitely generated.

Clearly (3) implies (4). Hence, assume (4). Suppose that dim $R > 1$. Then by Theorem A10 there exist two distinct minimal prime ideals P_1, P_2 of R. Let L be the complement of $P_1 \cup P_2$; then R_L has two maximal ideals and dim $R_L = 1$. By Theorem A7, Q/R_L decomposes into a direct sum. This contradicts the hypothesis, and so dim $R = 1$. By the corollary to Theorem 1 we know that every ring between R and Q is Noetherian and has dimension one. A repetition of the previous argument shows that they must all be local. Thus R is local, and if V is the integral closure of R, it follows that V is a discrete valuation ring.

Let N be the maximal ideal of V; then there exists $x \in V$ such that $N = Vx$.

Now if M is the maximal ideal of R, then V/N has D.C.C. over R/M by Theorem 1, and thus V/N is finitely generated over the field R/M. Let $v_1, \ldots, v_n$ be representatives of these generators in V, and let $R' = R[x, v_1, \ldots, v_n]$. By the above remarks R' is a Noetherian, local ring with maximal ideal M'. We have $V = VM' + R'$, and hence $M'(V/R') = V/R'$. But V/R' has D.C.C. over R' by Theorem 1, and so it follows from Proposition 1 that V/R' is divisible over R', *a fortiori* divisible over R. If we now assume condition (3), then $V = R'$, and so (1) is true.

Remarks. It is an open question whether condition (4) of the previous theorem is equivalent to the other three conditions. As the theorem shows, the finiteness of condition (2) is too strong to characterize local domains of dimension one. Now if $A \neq 0$ is any finitely generated module over a local ring R with maximal ideal M, then $MA \neq A$. If we replace condition (2) by this weaker condition on proper R-submodules of Q, then the following theorem shows that this does characterize local domains of dimension one.

THEOREM 3. *Let R be a Noetherian, local domain with maximal ideal M. Then* $\dim R = 1$ *if and only if $MS \neq S$ for every non-zero, reduced, torsion-free R-module S of finite rank.*

Proof. If $\dim R > 1$, take a non-zero prime ideal $P \neq M$. Then we have $MR_P = R_P$. Conversely, assume that $\dim R = 1$.

Case I: Rank $S = 1$. Suppose $MS = S$. Choose $x \neq 0 \in S$. Then $M(S/Rx) = S/Rx$. Since S/Rx has D.C.C. by Theorem 1, it follows from Proposition 1 that S/Rx is divisible. Let $a \neq 0 \in R$; then we have $S = aS + Rx$. Therefore, $S/aS \cong Rx/(aS \cap Rx)$ is finitely generated. However, $M(S/aS) = S/aS$; and so $S = aS$. Thus S is divisible. This contradicts the assumption that S is reduced, and thus $MS \neq S$.

Case II: Rank $S > 1$. Suppose rank $S = n$, and make the induction hypothesis that the theorem is true for modules of smaller rank. We note that for any R-module A, $A \otimes_R R/M \cong A/MA$. Now S has a reduced, torsion-free submodule T of rank $n - 1$ such that S/T is torsion-free of rank 1. We have an exact sequence:

$$\operatorname{Tor}_1^R(S/T, R/M) \to T \otimes_R R/M \to S \otimes_R R/M \to S/T \otimes_R R/M \to 0.$$

If S/T is reduced, then $S/T \otimes_R R/M \neq 0$ by Case I, and so we have $S \otimes_R R/M \neq 0$. On the other hand, if S/T is not reduced, then $S/T \cong Q$, and we have $\operatorname{Tor}_1^R(S/T, R/M) = 0 = S/T \otimes_R R/M$. Hence we have $T \otimes_R R/M \cong S \otimes_R R/M$; and since $T \otimes_R R/M \neq 0$ by induction, the proof of the theorem is complete.

2. $\operatorname{Ext}_R^1(Q, S) = 0$.

PROPOSITION 2. *Let R be a Noetherian, local domain with maximal ideal M, $E = E(R/M)$, and completion $\hat{R}$. Then the following are equivalent:*

(1) *If $\bar{I}$ is any non-zero ideal of $\bar{R}$, then $\bar{I} \cap R \neq 0$.*
(2) *E has no proper, faithful R-submodules.*
(3) *E has no proper, non-zero, divisible R-submodules.*

Proof.
(1) $\Rightarrow$ (2). Suppose that B is a proper, faithful R-submodule of E. By Theorem A7, B is also an $\bar{R}$-module. Let $\bar{I} = \{\bar{r} \in \bar{R} \mid \bar{r}B = 0\}$. Since $B \neq E$, $\bar{I} \neq 0$ by Theorem A2. However, since B is faithful as an R-module, $\bar{I} \cap R = 0$. This contradiction shows that E has no proper, faithful R-submodules.
(2) $\Rightarrow$ (3). This is trivial, since a divisible module is, *a fortiori*, faithful.
(3) $\Rightarrow$ (1). Suppose that $\bar{I}$ is an ideal of $\bar{R}$ such that $\bar{I} \cap R = 0$. Let $\bar{P}$ be an ideal of $\bar{R}$ such that $\bar{I} \subset \bar{P}$, and such that $\bar{P}$ is maximal with respect to the property that $\bar{P} \cap R = 0$. Since R is an integral domain, it is easily verified that $\bar{P}$ is a prime ideal of $\bar{R}$. Let B be the annihilator of $\bar{P}$ in E. Then by Theorem A1 B, considered as an $\bar{R}/\bar{P}$-module, is injective. Since $\bar{R}/\bar{P}$ is an integral domain, B is a divisible $\bar{R}/\bar{P}$-module. But $\bar{P} \cap R = 0$, and so B is divisible as an R-module. Hence by assumption $B = 0$. Thus $\bar{P} = 0$, and so $\bar{I} = 0$.

Two cases where the conditions of Proposition 2 are true are the following:
(i) *R is a complete domain.* For then (1) is trivially satisfied.
(ii) Dim $R = 1$ *and $\bar{R}$ is an integral domain.* For if $\bar{M}$ is the maximal ideal of $\bar{R}$, then $\bar{M} \cap R = M$. Since $\bar{M}$ is the only non-zero prime ideal of $\bar{R}$, condition (1) is fulfilled.

PROPOSITION 3. *Let R be a complete, Noetherian, local domain with maximal ideal M and $E = E(R/M)$. Then* dim $R = 1$ *if and only if every proper R-submodule of E is finitely generated.*

Proof. Suppose that dim $R = 1$, and let $B \neq 0$ be a proper submodule of E. Then $0(B)$ is an M-primary ideal, and so $0(B) = I_1 \cap \ldots \cap I_n$, where the I_j's are irreducible, M-primary ideals. By Theorem A1 there exist elements $x_1, \ldots, x_n \in E$ such that $I_j = 0(x_j)$. By Theorem A2,

$$B = \mathrm{Ann}_R(0(B)) = \mathrm{Ann}_R\left(\bigcap_{j=1}^{n} I_j\right) = \sum_{j=1}^{n} \mathrm{Ann}_R(I_j) = \sum_{j=1}^{n} Rx_j.$$

Thus B is finitely generated.

Conversely, suppose that every proper R-submodule of E is finitely generated. Let $I \neq 0$ be an ideal of R. Then $\mathrm{Ann}_E(I)$ is a proper submodule of E by Theorem A2. Hence there exist elements $x_1, \ldots, x_n \in E$ such that

$$\mathrm{Ann}_E(I) = \sum_{j=1}^{n} Rx_j.$$

Then by Theorem A2 again, we have

$$I = 0(\mathrm{Ann}_E(I)) = 0\left(\sum_{j=1}^{n} Rx_j\right) = \bigcap_{j=1}^{n} 0(x_j).$$

By Theorem A1 each $0(x_j)$ is an irreducible, M-primary ideal. Thus I is an M-primary ideal, Since I was an arbitrary non-zero ideal of R, it follows that M is the only non-zero, prime ideal of R, and so $\dim R = 1$.

PROPOSITION 4. *Let R be an arbitrary integral domain, and let $S \neq Q$ be a torsion-free R-module of rank one. Then the following are equivalent:*

(1) $\mathrm{Hom}_R(Q, Q/S) \cong Q$.

(2) $\mathrm{Ext}_R^1(Q, S) = 0$.

(3) $S \cong \mathrm{Ext}_R^1(K, S)$.

If any of these conditions hold, then Q/S is an indecomposable R-module.

Proof. Since $\mathrm{Hom}_R(Q, S) = 0$ and $\mathrm{Hom}_R(Q, Q) \cong Q$, we have an exact sequence

$$0 \to Q \to \mathrm{Hom}_R(Q, Q/S) \to \mathrm{Ext}_R^1(Q, S) \to 0.$$

From this sequence it follows immediately that (1) and (2) are equivalent. Since $\mathrm{Hom}_R(R, S) \cong S$, we also have an exact sequence:

$$0 \to S \to \mathrm{Ext}_R^1(K, S) \to \mathrm{Ext}_R^1(Q, S) \to 0.$$

Since $\mathrm{Ext}_R^1(Q, S)$ is a Q-module, the equivalence of (2) and (3) follows readily from this sequence and considerations of rank.

Now assume that $\mathrm{Ext}_R^1(Q, S) = 0$, and suppose that we have a direct sum decomposition: $Q/S = T/S \oplus U/S$, where T, U are proper R-submodules of Q that contain S. Then $T + U = Q$ and $T \cap U = S$. Thus we have an exact sequence:

$$0 \to S \to T \oplus U \to Q \to 0.$$

Since $\mathrm{Ext}_R^1(Q, S) = 0$, we have $T \oplus U \cong S \oplus Q$. Thus we have a non-zero projection of Q into one of the modules T or U. This is impossible, since both T and U are reduced. Therefore, Q/S is indecomposable.

Remark. We note that if R is any integral domain, then by Theorem A5 we have $\mathrm{Ext}_R^1(Q, R) = 0$ if and only if $\mathrm{Hom}_R(K, K) \cong R$.

COROLLARY. *Let R be an arbitrary integral domain, and V a ring between R and Q. Let T be a torsion-free V-module of rank one. Then* $\mathrm{Ext}_V^1(Q, T) = 0$ *if and only if* $\mathrm{Ext}_R^1(Q, T) = 0$.

Proof. Using Theorem A11 we have

$$\mathrm{Hom}_V(Q, Q/T) \cong \mathrm{Hom}_V(Q \otimes_R V, Q/T) \cong \mathrm{Hom}_R(Q, \mathrm{Hom}_V(V, Q/T)) \cong \mathrm{Hom}_R(Q, Q/T).$$

Then the corollary follows from Proposition 4.

PROPOSITION 5. *Let R be a Noetherian, local domain with maximal ideal M and $E = E(R/M)$. Then R is complete and* $\dim R = 1$ *if and only if* $Q \cong \mathrm{Hom}_R(Q, E)$.

Proof. If A is an R-module, we will denote by A^* the module $A^* = \mathrm{Hom}_R(A, E)$.

Suppose that R is complete and $\dim R = 1$. We have a commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 \to & R & \to & Q & \to & K & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & R^{**} & \to & Q^{**} & \to & K^{**} & \to 0. \end{array}$$

Since K has D.C.C. by Theorem 1, we have by Theorem A2 that the maps $R \to R^{**}$ and $K \to K^{**}$ are isomorphisms. It follows that the map $Q \to Q^{**}$ is an isomorphism. Since Q^* is a Q-module, it follows from consideration of rank that $Q \cong Q^*$.

Now suppose that $Q \cong Q^*$. Now we have $R^{**} \subset Q^{**}$; but since $Q \cong Q^{**}$ by assumption and $R^{**} \cong \hat{R}$, the completion of R, by Theorem A1, we actually have $\hat{R} \subset Q$. However, as is well known, $\hat{R} \cap Q = R$. Thus $R = \hat{R}$; that is, R is complete.

Suppose $\dim R > 1$. Then by Theorem A10 we can find two distinct minimal prime ideals P_1, P_2 of R. Let L be the complement of $P_1 \cup P_2$; and let $C = \mathrm{Hom}_R(R_L, E)$. Then C is an injective R_L-module (**2**, Prop. 2.6.1a). Using Theorem A11 we have

$$\begin{aligned} \mathrm{Hom}_{R_L}(C, C) = \mathrm{Hom}_{R_L}(C, \mathrm{Hom}_R(R_L, E) &\cong \mathrm{Hom}_R(C \otimes_{R_L} R_L, E) \\ &\cong \mathrm{Hom}_R(C, E) \cong {R_L}^{**}. \end{aligned}$$

Since ${R_L}^{**} \subset Q^{**} \cong Q$, ${R_L}^{**}$ is R_L-indecomposable. Therefore, C is an indecomposable injective R_L-module. Now we have

$$\mathrm{Hom}_{R_L}(R_L/P_1R_L, C) \cong \mathrm{Hom}_R(R_L/P_1R_L, E) \neq 0.$$

Thus, since P_1R_L is a maximal ideal of R_L, C has an element of order P_1R_L; and thus C is the injective envelope over R_L of R_L/P_1R_L. Similarly, C is the injective envelope over R_L of R_L/P_2R_L. Thus by Theorem A1, $P_1R_L = P_2R_L$; and so $P_1 = P_2$. Since P_1, P_2 were chosen distinct, this contradiction shows that $\dim R = 1$.

The unusual strength of the condition $Q \cong Q^*$ will be investigated further in Proposition 6, but first we need a lemma.

LEMMA 1. *Let V be a valuation ring with quotient field Q, maximal ideal N, and $C = E(V/N)$. Then V is a maximal valuation ring if and only if* $\mathrm{Hom}_V(C, C) \cong V$; *and in this case $C \cong Q/N$.*

Proof. If V is a maximal valuation ring, then $\mathrm{Hom}_V(C, C) \cong V$ by Theorem A3. Conversely, assume that $\mathrm{Hom}_V(C, C) \cong V$. We will show first that $C \cong Q/N$. Now C is the injective envelope of Q/N by (**8**, Prop. 1); and, therefore, $Q/N \subset C$. Let $x \neq 0 \in C$, and choose $a \neq 0 \in 0(x)$. Let $y = 1/a + N \in Q/N$; then $0(y) = aN \subset 0(x)$. Hence we have a map $f: Ry \to Rx$ such that $f(y) = x$. Then f can be extended to an endomorphism

$g: C \to C$. But g is multiplication by an element $v \in V$. Thus $x = g(y) = vy \in Q/N$. Hence $C \subset Q/N$, and so $C = Q/N$.

We next show that Q/V is injective. Let A be any V-module. Since w. gl. dim. $V = 1$ by **(4**, Theorem 2**)**, we have $\mathrm{Tor}_2^V(A, V/N) = 0$. Thus, since $V/N \cong \mathrm{Hom}_V(V/N, C)$, we have by Theorem A11,

$$\mathrm{Ext}_V^2(A, V/N) \cong \mathrm{Ext}_V^2(A, \mathrm{Hom}_V(V/N, C)) \cong \mathrm{Hom}_V(\mathrm{Tor}_2^V(A, V/N), C) = 0.$$

Hence the injective dimension of V/N is one. Therefore, since $Q/N \cong C$ is injective, we conclude from the exact sequence:

$$0 \to V/N \to Q/N \to Q/V \to 0$$

that Q/V is injective. Thus by Theorem A3, V is an almost maximal valuation ring. Since $\mathrm{Hom}_V(Q/N, Q/N) \cong V$, it follows from Theorem A3 that V is a maximal valuation ring.

PROPOSITION 6. *Let R be a local domain (not necessarily Noetherian) with maximal ideal M and $E = E(R/M)$. Then the following are equivalent:*

(1) $Q \cong \mathrm{Hom}_R(Q, E)$.

(2) *If V is any valuation ring between R and Q, then V is a maximal valuation ring; and if $C = \mathrm{Hom}_R(V, E)$, then C is the injective envelope over V of V/N, where N is the maximal ideal of V.*

Proof.

(1) $\Rightarrow$ (2). If A is any R-module, we let $A^* = \mathrm{Hom}_R(A, E)$. Let V be any valuation ring between R and Q, and let N be the maximal ideal of V. Let $C = \mathrm{Hom}_R(V, E)$ and $\Lambda = \mathrm{Hom}_V(C, C)$. Then by a repetition of the type of argument used in the second part of Proposition 5 we can show that C is the injective envelope over V of V/N, and that $\Lambda \cong V^{**}$. Thus $V \subset \Lambda \subset Q$. By **(6**, Prop. 2.6**)** Λ is a local ring; denote its maximal ideal by P. Then to prove that $V = \Lambda$, it is sufficient to prove that $N \subset P$. However, if $v \in N$, then there exists $x \in C$ such that $vx = 0$, and so $v \in P$. Thus $V = \Lambda$. It now follows from Lemma 1 that V is a maximal valuation ring.

(2) $\Rightarrow$ (1). Let V be a valuation ring between R and Q. Let N be the maximal ideal of V and $C = \mathrm{Hom}_R(V, E)$. Then by assumption V is a maximal valuation ring, and C is the injective envelope over V of V/N. By Theorem A3 we have $C \cong Q/N$. Now $\mathrm{Ext}_V^1(Q, N) = 0$ by Theorem A4, and so $\mathrm{Hom}_V(Q, Q/N) \cong Q$ by Proposition 4. Thus, using Theorem A11 we have

$$\mathrm{Hom}_R(Q, E) \cong \mathrm{Hom}_R(Q \otimes_V V, E) \cong \mathrm{Hom}_V(Q, \mathrm{Hom}_R(V, E)) \cong \mathrm{Hom}_V(Q, Q/N) \cong Q.$$

We note that if R is a Noetherian, local domain, then it follows from Propositions 5 and 6 that R is complete and $\dim R = 1$ if and only if condition (2) of Proposition 6 is satisfied.

THEOREM 4. *Let R be a Noetherian integral domain. Then the following are equivalent:*

(1) *R is a complete local ring and* $\dim R = 1$.

(2) *Every proper R-submodule of Q is finitely generated and* $\mathrm{Hom}_R(K, K) \cong R$.

(3) $\mathrm{Ext}_R^1(Q/S, R) \neq 0$ *for every proper, non-zero, R-submodule S of Q and* $\mathrm{Ext}_R^1(Q, R) = 0$.

(4) $\mathrm{Ext}_R^1(Q, S) = 0$ *for every R-submodule S of Q.*

(5) *Every R-homomorphic image of Q is indecomposable and* $\mathrm{Hom}_R(K, K) \cong R$.

(6) *R is a local ring,* $\dim R = 1$, *and* $\mathrm{Hom}_R(K, K) \cong R$.

Proof.

(1) $\Rightarrow$ (2). We will denote the maximal ideal of R by M, and let $E = E(R/M)$. If A is any R-module, we let $A^* = \mathrm{Hom}_R(A, E)$. By Proposition 5 we have $Q \cong Q^*$; and, of course, we have $R^* = E$. Thus we have an exact sequence:

$$0 \to K^* \to Q \to E \to 0,$$

and so $E \cong Q/K^*$. By Theorem 1, K has D.C.C. Therefore, by Theorem A2, K^* is finitely generated. Hence, since K^* is an R-submodule of Q, K^* is isomorphic to an ideal I of R. Thus we have $E \cong Q/K^* \cong Q/I$, and so K is a homomorphic image of E. It now follows from Proposition 3 that every proper submodule of K is finitely generated. It is an immediate consequence that every proper R-submodule of Q is finitely generated. By Theorem A5 we have $\mathrm{Hom}_R(K, K) \cong R$.

(2) $\Leftrightarrow$ (3). Let S be a proper, non-zero R-submodule of Q. Then we have an exact sequence:

$$0 \to \mathrm{Hom}_R(S, R) \to \mathrm{Ext}_R^1(Q/S, R) \to \mathrm{Ext}_R(Q, R).$$

Since $\mathrm{Hom}_R(K, K) \cong R$ is equivalent to $\mathrm{Ext}_R^1(Q, R) = 0$ by Theorem A5, we have, assuming either (2) or (3), that $\mathrm{Hom}_R(S, R) \cong \mathrm{Ext}_R^1(Q/S, R)$. The equivalence of (2) and (3) now follows from the fact that $\mathrm{Hom}_R(S, R) \neq 0$ if and only if S is finitely generated.

(2) $\Rightarrow$ (4). This follows immediately from Theorem A5.

(4) $\Rightarrow$ (5). This is an immediate consequence of Proposition 4 and Theorem A5.

(5) $\Rightarrow$ (6). This follows directly from Theorem 2.

(6) $\Rightarrow$ (1). If $\hat{R}$ is the completion of R, we can define an operation of $\hat{R}$ on K as follows. Let $x \in K$ and $\hat{r} \in \hat{R}$. Then there exists a Cauchy sequence $\{r_n\}$ of elements $r_n \in R$ such that $r_n \to \hat{r}$. Let M be the maximal ideal of R. Then there exists an integer $k > 0$ such that $M^k x = 0$, and such that $r_n - r_m \in M^k$, whenever $n, m \geqslant k$. We define $\hat{r}x = r_k x$. It is easily verified that this definition makes K into an $\hat{R}$-module. Suppose $\hat{r}K = 0$. Let $a \neq 0 \in M$. Then there exists an integer $N > 0$ such that $r_n \in Ra$ for all $n > N$; hence $r_n \to 0$, and so $\hat{r} = 0$. Thus we have $\hat{R} \subset \mathrm{Hom}_R(K, K)$. Since $\mathrm{Hom}_R(K, K) \cong R$ by assumption, it follows that R is complete.

We note that if R is a complete, Noetherian, local domain, then by Theorem A5 we may omit the hypotheses $\mathrm{Hom}_R(K, K) \cong R$ and $\mathrm{Ext}_R^1(Q, R) = 0$ wherever they appear in the statement of Theorem 4. As a corollary of Theorem 4 we obtain a special case of a theorem of M. Nagata **(11**, Th. 7**)** (see also D. G. Northcott **(12**, Prop. 4**)**).

COROLLARY 1 (*special case of a theorem of M. Nagata*). *Let R be a complete, Noetherian, local domain such that* $\dim R = 1$. *Then the integral closure of R is a complete, discrete, valuation ring that is finitely generated over R.*

Proof. This follows immediately from Theorem 4 and the corollary to Proposition 4.

COROLLARY 2. *Let R be a complete, Noetherian, local domain with maximal ideal M and $E = E(R/M)$. Then* $\dim R = 1$ *if and only if* $\mathrm{Ext}_R^1(E, R) \neq 0$.

Proof. Suppose $\dim R = 1$. Now by Proposition 2 there exists a proper, non-zero R-submodule T of Q such that $E \cong Q/T$. Hence $\mathrm{Ext}_R^1(E, R) \neq 0$ by Theorem 4. Conversely, assume that $\mathrm{Ext}_R^1(E, R) \neq 0$. Let S be a proper, non-zero, R-submodule of Q. Then by Proposition 2 there exists an R-submodule T of Q such that $S \subset T$ and such that the following sequence is exact:

$$0 \to T/S \to Q/S \to E \to 0.$$

Since $\mathrm{Hom}_R(T/S, R) = 0$, we have $\mathrm{Ext}_R^1(E, R) \subset \mathrm{Ext}_R^1(Q/S, R)$, and thus $\mathrm{Ext}_R^1(Q/S, R) \neq 0$. Hence $\dim R = 1$ by Theorem 4.

COROLLARY 3. *Let R be a complete, Noetherian, local domain such that* $\dim R = 1$. *Then every torsion-free R-module of finite rank is a direct sum of a finite number of copies of Q and of a finitely generated R-module.*

Proof. Let S be a torsion-free R-module of finite rank. To prove the corollary it is sufficient to assume that S is reduced, and then prove that S is finitely generated. We proceed by induction on rank S. If rank $S = 1$, then S is finitely generated by Theorem 4. Hence assume that rank $S = n > 1$, and assume the theorem true for modules of smaller rank. Now S has a submodule T of rank $n - 1$ such that S/T is torsion-free of rank 1. Thus T is finitely generated by the induction hypothesis. If S/T is not reduced, then $S/T \cong Q$, and we have $\mathrm{Ext}_R^1(S/T, T) = 0$ by Theorem 4. Thus S/T is a direct summand of S, which contradicts the fact that S is reduced. Hence S/T is reduced, and so S/T is finitely generated. Thus S is finitely generated.

COROLLARY 4. *Let R be a complete, Noetherian, local domain such that* $\dim R = 1$. *Let S be a torsion-free R-module of finite rank. Then* $\mathrm{hd}_R S < \infty$ *if and only if S is a direct sum of a free R-module and copies of Q. And in this case,* $\mathrm{hd}_R S \leqslant 1$.

Proof. By Corollary 3, $S = T \oplus D$, where T is a finitely generated submodule of S, and D is a direct sum of copies of Q. By Theorem A6, $\mathrm{hd}_R D \leqslant 1$.

Thus, $\mathrm{hd}_R S < \infty$ if and only if $\mathrm{hd}_R T < \infty$. Since f. gl. dim. $R = 1$, and since T is a submodule of a free R-module, it follows that $\mathrm{hd}_R T < \infty$ if and only if T is free.

3. K is injective.

THEOREM 5. *Let R be a Noetherian integral domain such that K is injective. Then* $\dim R = 1$, *and* $K = \sum \oplus E(R/M_\alpha)$, *where M_α ranges over all of the maximal ideals of R, and each $E(R/M_\alpha)$ appears exactly once.*

Proof.

Case I: R is a local ring with maximal ideal M. Let A be any indecomposable, direct summand of K. Then by Theorem A1, $A = E(R/P)$, where P is a non-zero prime ideal of R. Now P^{-1}/R is contained in K, and is a finitely generated R/P-module. By Theorem A1, a copy of the quotient field of R/P is contained in $A \cap P^{-1}/R$. Thus the quotient field of R/P is finitely generated over R/P, and so is equal to R/P. Thus P is a maximal ideal, and hence $P = M$. Therefore, by Theorem A1, we see that $K = \sum \oplus E_\beta$, where each $E_\beta = E(R/M) = E$. Now

$$E \cong \mathrm{Tor}_1^R(K, E) \cong \sum \oplus \mathrm{Tor}_1^R(E_\beta, E).$$

Since E is indecomposable, $K \cong E$. Let P' be any non-zero, prime ideal of R. Then we have an exact sequence:

$$0 \to R_{P'} \to Q \to E \otimes_R R_{P'} \to 0.$$

Since $R_{P'} \neq Q$, we have $E \otimes_R R_{P'} \neq 0$; and thus $P' = M$. Hence $\dim R = 1$.

Case II: R is an arbitrary Noetherian domain. Let M be any maximal ideal of R. Then by **(2**, Ch. 6, Ex. 11**)**, $K_M = K \otimes_R R_M \cong Q/R_M$ is an injective R_M-module. Hence by Case I, $\dim R_M = 1$. Thus $\dim R = 1$. Now by Theorem A7, $K = \sum \oplus K_{M_\alpha}$, where M_α ranges over all of the maximal ideals of R. By Case I,

$$K_{M_\alpha} \cong E(R_{M_\alpha}/M_\alpha R_{M_\alpha}) = E(R/M_\alpha).$$

Thus each $E(R/M_\alpha)$ appears exactly once.

The following corollary is a generalization of a theorem of R. J. Nunke **(13**, Cor. 7.9**)**.

COROLLARY. *Let R be a Noetherian integral domain such that K is injective. Then R is a complete local ring and* $\dim R = 1$ *if and only if* $\mathrm{Ext}_R^1(K, R) \cong R$.

Proof. Suppose $\mathrm{Ext}_R^1(K, R) \cong R$. Then by Proposition 4 and Theorem A5, we have $\mathrm{Hom}_R(K, K) \cong R$. Thus K is indecomposable, and so by Theorem 5 R is a local ring and $\dim R = 1$. By Theorem 4, R is complete. Conversely, if R is any complete local domain, then $\mathrm{Ext}_R^1(Q, R) = 0$ by Theorem A5. Consequently, $\mathrm{Ext}_R^1(K, R) \cong R$ by Proposition 4.

THEOREM 6. *Let R be a Noetherian, local domain such that* dim $R = 1$*; and such that $\bar{R}$, the completion of R, is an integral domain. Let M be the maximal ideal of R and $E = E(R/M)$. Then the following are equivalent:*

(1) *K is injective.*

(2) $\mathrm{Hd}_R E = 1$.

(3) *R has an irreducible, principal ideal.*

(4) $M^{-1}/R \cong R/M$.

(5) *M^{-1} is generated by at most two elements.*

If any of the above conditions hold, then every proper R-submodule of Q is finitely generated; and every principal ideal of R is irreducible.

Proof.

(1) $\Rightarrow$ (2). By Theorem 5 we have $K \cong E$. By Theorem A6, $\mathrm{hd}_R K = 1$.

(2) $\Rightarrow$ (1). By assumption we have an exact sequence:

$$0 \to F_1 \to F_0 \to E \to 0,$$

where F_1, F_0 are free R-modules. Let $F_1^* = \mathrm{Hom}_R(F_1, E)$ and $F_0^* = \mathrm{Hom}_R(F_0, E)$. Since $\mathrm{Hom}_R(E, E) \cong \bar{R}$ by Theorem A1, we have an exact sequence:

$$0 \to \bar{R} \to F_0^* \to F_1^* \to 0.$$

Now F_0^*, F_1^* are injective R-modules, and so the injective dimension of $\bar{R}$, considered as an R-module, is one. Since $Q \otimes_R \bar{R}$ is R-injective, we conclude from the exact sequence:

$$0 \to \bar{R} \to Q \otimes_R \bar{R} \to K \otimes_R \bar{R} \to 0$$

that $K \otimes_R \bar{R}$ is R-injective. By Theorem 1, K has D.C.C., and so by Theorem A7 we have $K \cong K \otimes_R \bar{R}$. Thus K is injective.

(1) $\Rightarrow$ (3). Let $s \neq 0 \in M$, and let $x = 1/s + R$, $x \in K$. Then $0(x) = sR$. Hence, since $K \cong E$, sR is irreducible by Theorem A1. Thus every principal ideal of R is irreducible.

(3) $\Rightarrow$ (1). Suppose that $s \neq 0 \in M$, and sR is an irreducible ideal of R. By Theorem A1 there exists $x_1 \neq 0 \in E$ such that $0(x_1) = sR$. Since E is divisible, we can find $x_2 \in E$ such that $sx_2 = x_1$, and $x_3 \in E$ such that $sx_3 = x_2$, etc. Let L be the multiplicative system consisting of the powers of s. Then $R_L = Q$, and so K is generated by the set $\{1/s^k + R\}$. Define $f: K \to E$ by $f(a/s^k + R) = ax_k$, $a \in R$. It is easily verified that f is a monomorphism. Thus $K \subset E$, and so by Proposition 2, $K = E$.

(1) $\Rightarrow$ (4). We have $\mathrm{Hom}_R(R/M, K) \cong \mathrm{Ann}_K(R/M) = M^{-1}/R$. If $K \cong E$, then $\mathrm{Hom}_R(R/M, K) \cong R/M$ by Theorem A1. Therefore, $M^{-1}/R \cong R/M$.

(4) $\Rightarrow$ (1). M^{-1}/R is the socle of K. Since R has dimension one, it is easily seen that K is an essential extension of M^{-1}/R. Thus $E(K) = E(M^{-1}/R)$. Hence, if $M^{-1}/R \cong R/M$, then $E(K) = E$. But then $K = E$ by Proposition 2.

(4) $\Rightarrow$ (5). If $M^{-1}/R \cong R/M$, take $u \in M^{-1}$, $u \notin R$. Then M^{-1} will be generated by $1, u$.

(5) $\Rightarrow$ (4). Suppose M^{-1} is generated by at most two elements. If $MM^{-1} = R$, then M is projective, hence principal, and so we have $M^{-1}/R \cong R/M$. Hence assume $MM^{-1} \neq R$. Then $MM^{-1} = M$, and we have an exact sequence:

$$0 \to R/M \to M^{-1}/M \to M^{-1}/R \to 0$$

over the field R/M. Therefore, the sequence splits, and we have $M^{-1}/M \cong R/M \oplus M^{-1}/R$. Now M^{-1}/M is a vector space of dimension two over R/M, and so M^{-1}/R is one dimensional over R/M. Thus $M^{-1}/R \cong R/M$.

Now assume that K is injective. Let S be a proper, non-zero R-submodule of Q. We can assume that $R \subset S$. Then $S/R \subset K$, and $S/R \neq K$. By Proposition 2, S/R is not a faithful submodule of K. Hence there exists $a \neq 0 \in R$ such that $a(S/R) = 0$. Therefore, $aS \subset R$, and so S is a finitely generated R-module.

COROLLARY. *Let R be a Noetherian, local domain such that* dim $R = 1$ *and $\bar{R}$ is an integral domain. If M is generated by at most two elements, then K is injective.*

Proof. By Theorem 6 it will be sufficient to prove that M^{-1} is generated by at most two elements. Let $u_1, u_2 \in M$ be the generators of M. We can assume that $u_2 \notin u_1R$. There exists an integer $n > 0$ such that $u_2^n \notin u_1R$, but $u_2^{n+1} \in u_1R$. Let B be the R-module generated by 1, u_2^n/u_1. Then $B \subset M^{-1}$. Conversely, let $x \in M^{-1}$. Then $x = b/u_1$, where $b \in M$. If $b \in u_1R$, then $x \in B$; hence assume that $b \notin u_1R$. Since $Ru_1 + Ru_2^k = Ru_1 + M^k$, and since

$$\bigcap_k (Ru_1 + M^k) = Ru_1,$$

there exists an integer $k > 0$ such that $b \in Ru_1 + Ru_2^k$, but $b \notin Ru_1 + Ru_2^{k+1}$. Thus $b = ru_1 + su_2^k$, where $r \in R$ and $s \in R - M$. Hence $x = b/u_1 = r + s(u_2^k/u_1)$; and since s is a unit, $(u_2^k/u_1) \in M^{-1}$. Hence $u_2^{k+1} \in u_1R$, which means that $k \geqslant n$, and so $x \in B$. Thus $M^{-1} = B$, and so M^{-1} is generated by at most two elements.

PROPOSITION 7. *Let R be an integral domain and $S \neq 0$ a torsion-free R-module such that S, and every factor module of S by a cyclic submodule, is indecomposable. Let A be an extension of R by S. Then, if A is not the split extension, A is an indecomposable, torsion-free R-module and rank A = rank $S + 1$.*

Proof. Suppose that A decomposes into a direct sum: $A = A_1 \oplus A_2$, where A_1, A_2 are non-zero submodules of A. Now there exist $x_1 \in A_1$, $x_2 \in A_2$ such that, if $x = x_1 + x_2$, then $A/Rx \cong S$. We must prove that Rx is a direct summand of A. Let $B = Rx_1 \oplus Rx_2$; then $B = Rx + Rx_2$, and so B/Rx is cyclic. Since $B/Rx \subset A/Rx$, B/Rx is torsion-free. Thus Rx is a direct summand of B, and $B = Rx \oplus T$, where T is a submodule of B such that either $T = 0$ or $T \cong R$.

Now we have $A_1/Rx_1 \oplus A_2/Rx_2 \cong A/B \cong (A/Rx)/(B/Rx) \cong S/T'$, where T' is a submodule of S such that $T' \cong T$. Since S/T' is indecomposable by

assumption, we can assume that $A_2/Rx_2 = 0$; that is, $A_2 = Rx_2$. If $y \in A_2$, then $y = rx_2 = rx - rx_1$, where $r \in R$. Thus $A_2 \subset Rx + A_1$, and so $A = Rx + A_1$. Suppose $z \neq 0 \in Rx \cap A_1$. Then $z = rx$, $r \in R$, and $z = y_1 \in A_1$. Hence $rx_2 = 0$. Since $z \neq 0$, we have $r \neq 0$, and thus $x_2 = 0$. But then $A_2 = 0$, which is a contradiction to assumption. Hence $Rx \cap A_1 = 0$, and so $A = Rx \oplus A_1$. Thus Rx is a direct summand of A which makes A the split extension.

We now list four examples of integral domains that have indecomposable, torsion-free modules of rank two.

(1) *R is an integral domain such that K is indecomposable and* $\mathrm{Hom}_R(K, K) \neq R$.

Proof. By Theorem A5 we have $\mathrm{Ext}_R^1(Q, R) \neq 0$. The conclusion follows from Proposition 7 and the fact that there is an extension of R by Q that is not the split extension.

(2) *R is a Noetherian, local domain such that K is injective, but R is not complete.*

Proof. By Theorem 5 we have $K \cong E$, and so K is indecomposable. Since R is not complete, $\mathrm{Hom}_R(K, K) \neq R$ by Theorem A1. The conclusion now follows from example (1).

(3) *R is a Noetherian domain such that every R-homomorphic image of Q is indecomposable, but R is not complete.*

Proof. Since R is not complete, we have by Theorem 4 that $\mathrm{Hom}_R(K, K) \neq R$. The conclusion now follows from example (1).

(4) *R is an integral domain such that every principal ideal of R is irreducible, but K is not injective.*

Proof. Since K is not injective, there exists a non-zero ideal of R such that $\mathrm{Ext}_R^2(R/I, R) \neq 0$. Hence $\mathrm{Ext}_R^1(I, R) \neq 0$, and so there exist extensions of R by I that are not split. Since every principal ideal of R is irreducible, every factor module of I by a cyclic submodule is indecomposable. The conclusion now follows from Proposition 7.

COROLLARY. *Let R be a valuation ring. Then R is a maximal valuation ring if and only if every torsion-free R-module of rank two decomposes into a direct sum of two R-modules of rank one.*

Proof. If R is maximal, see **(5**, Th. 12**)** or Theorem A4. Conversely, assume that every torsion-free R-module of rank two is decomposable. By the preceding example (4) we have that K is injective. Hence by Theorem A3, R is almost maximal. By the preceding example (1) we have that $\mathrm{Hom}_R(K, K) \cong R$. Thus R is a maximal valuation ring by Theorem A4.

Appendix

THEOREM A1. *Let R be a commutative, Noetherian ring. Then:*

(1) *Every injective R-module is a direct sum of indecomposable injective R-modules.*

(2) *There is a one-to-one correspondence between the prime ideals P of R and the indecomposable injective R-modules given by $P \leftrightarrow E(R/P)$.*

(3) *If P is a prime ideal of R, then I is an irreducible, P-primary ideal of R if and only if there exists an element $x \neq 0 \in E(R/P)$ such that $I = 0(x)$.*

(4) $\mathrm{Hom}_R(E(R/P), E(R/P)) \cong \hat{R}_P$, *where $\hat{R}_P$ is the completion of R_P.*

(5) *If I is any ideal of R contained in the prime ideal P, then $\mathrm{Hom}_R(R/I, E(R/P))$ is the injective envelope over R/I of $(R/I)/(P/I)$. Thus, in particular, $\mathrm{Hom}_R(R/P, E(R/P))$ is isomorphic over R/P to the quotient field of R/P.*

Proof. (7, Th. 2.5), (7, Prop. 3.1), (7, Lemma 3.2), (7, Th. 3.4), (7, Th. 3.7).

THEOREM A2. *Let R be a complete, Noetherian, local ring with maximal ideal M and $E = E(R/M)$. Then there is a one-to-one, lattice-order inverting correspondence between the ideals I of R and the submodules A of E given by:*

$$I \leftrightarrow \mathrm{Ann}_R(I) \qquad \text{and} \qquad A \leftrightarrow 0(A),$$

such that $I = 0(\mathrm{Ann}_R(I))$ and $A = \mathrm{Ann}_R(0(A))$. If B is a finitely generated R-module (resp. has D.C.C.), *then $\mathrm{Hom}_R(B, E)$ has* D.C.C. *(resp. is finitely generated); and we have $\mathrm{Hom}_R(\mathrm{Hom}_R(B, E), E) \cong B$.*

Proof. (7, Th. 4.2), (7, Cor. 4.3).

THEOREM A3. *Let V be a valuation ring with maximal ideal N and quotient field Q. Then V is almost maximal if and only if Q/V is injective. If V is almost maximal, then V is maximal if and only if $\mathrm{Hom}_V(Q/N, Q/N) \cong V$. And if V is almost maximal, Q/N is the injective envelope of V/N.*

Proof. (8, Th. 4), (8, Lemma 7), (8, Th. 9).

THEOREM A4. *Let R be an integral domain. Then the following are equivalent:*

(1) *R is a maximal valuation ring.*

(2) *R is an almost maximal valuation ring and $\mathrm{Hom}_R(K, K) \cong R$.*

(3) $\mathrm{Ext}_R^1(A, S) = 0$ *for every torsion-free R-module A, and every torsion-free R-module S of rank one.*

Proof. (8, Th. 9).

THEOREM A5. *Let R be an integral domain. Then $\mathrm{Ext}_R^1(Q, R) = 0$ if and only if $\mathrm{Hom}_R(K, K) \cong R$; and in this case $\mathrm{Ext}_R^1(Q, S) = 0$ for every finitely generated R-submodule S of Q. If R is a complete, Noetherian, local domain, then the above conditions are satisfied.*

Proof. (8, Lemma 6), (7, Th. 4.2), (2, Prop. 6.5.1).

THEOREM A6. *Let R be a Noetherian integral domain of dimension one. Then*

(1) $\mathrm{Hd}_R Q = 1$.

(2) *Every divisible R-module is a homomorphic image of an injectiv R-module.*

Proof. **(9**, Lemma 3.2**)**, **(9**, Th. 3.3**)**.

THEOREM A7. *Let R be a commutative, Noetherian ring, and A an R-module such that if* $x \neq 0 \in A$, *then every prime ideal that belongs to* $0(x)$ *is maximal. Then:*

(1) $A = \sum \oplus A_{M_\alpha}$, *where* M_α *ranges over the maximal ideals of R.*

(2) $A_{M_\alpha} = A \otimes_R \bar{R}_{M_\alpha}$, *where* $\bar{R}_{M_\alpha}$ *is the completion of* R_{M_α}.

(3) A_{M_α} *is finitely generated (resp. has* D.C.C.*) over* R_{M_α} *if and only if the same is true over R.*

Proof. **(9**, Lemma 3.1**)** and **(10)**.

THEOREM A8. *Let A be a module over a Noetherian ring R. Then A has* D.C.C. *if and only if A is contained in a finite direct sum of modules of the form* $E(R/M_\alpha)$, *where* M_α *is a maximal ideal of R.*

Proof. **(7**, Cor. 4.3**)** and **(10)**.

THEOREM A9. *Let A be a module with* D.C.C. *over a commutative, Noetherian ring R. Let I be an ideal of R. Then* $IA = A$ *if and only if there exists an element* $r \in I$ *such that* $rA = A$.

Proof. **(10)**.

THEOREM A10. *Let R be a commutative, Noetherian ring such that* $\dim R \geqslant 2$. *Then R has an infinite number of prime ideals of rank one.*

Proof. **(1**, Prop. 2.6**)**.

THEOREM A11. *Let R, S be rings and consider the situation described by the symbol* $(A_R, {}_RB_S, C_S)$. *Then there is a natural isomorphism:*

$$\mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C)) \cong \mathrm{Hom}_S(A \otimes_R B, C).$$

If C is S-injective, then we have an isomorphism:

$$\mathrm{Ext}_R(A, \mathrm{Hom}_S(B, C)) \cong \mathrm{Hom}_S(\mathrm{Tor}^R(A, B), C).$$

Proof. **(2**, Prop. 2.5.2'**)**, **(2**, Prop. 6.5.1**)**.

REFERENCES

1. M. Auslander and D. Buchsbaum, *Homological dimension in Noetherian rings II*, Trans. Amer. Math. Soc., *88* (1958), 194–206.
2. H. Cartan and S. Eilenberg, *Homological algebra* (Princeton University Press, 1956).
3. I. S. Cohen, *Commutative rings with restricted minimum condition*, Duke Math J., *17* (1950), 27–42.

4. A. Hattori, *On Prufer rings*, J. Math. Soc. Japan, *9* (1957), 381–385.
5. I. Kaplansky, *Modules over Dedekind rings and valuation rings*, Trans. Amer. Math. Soc., *72* (1952), 327–340.
6. I. Kaplansky, *Infinite abelian groups* (University of Michigan Press, 1954).
7. E. Matlis, *Injective modules over Noetherian rings*, Pacific J. Math., *8* (1958), 511–528.
8. E. Matlis, *Injective modules over Prufer rings*, Nagoya Math. J., *15* (1959), 57–69.
9. E. Matlis, *Divisible modules*, Proc. Amer. Math. Soc., *11* (1960), 385–391.
10. E. Matlis, *Modules with D.C.C.*, Trans. Amer. Math. Soc. *97* (1960), 495–508.
11. M. Nagata, *Some remarks on local rings II*, Memoirs of the College of Science, University of Kyoto, Series A, *28*, Mathematics No. 2 (1953), 109–120.
12. D. G. Northcott, *General theory of one dimensional local rings*, Proc. Glasgow Math. Assoc., *2* (1956), 159–169.
13. R. J. Nunke, *Modules of extensions over Dedekind rings*, Illnois J. Math., *3* (1959), 222–240.

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A NOTE ON INDUCED MODULES

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1. Introduction. In this paper, A denotes a ring with an identity element 1, and B a subring of A containing 1 such that B satisfies the left and right minimum conditions, and A is a finitely generated left and right B-module. The identity element 1 is required to act as the identity operator on all modules which we shall consider. For any left B-module V, there is a standard construction of a left A-module which is, roughly speaking, the smallest A-module containing V. Namely, we form the tensor product group $A \otimes_B V$, and define the module operations in this group according to the rule

$$a\Big(\sum a_i \otimes v_i\Big) = \sum aa_i \otimes v_i, \qquad a, a_i \in A, \qquad v_i \in V. \tag{1}$$

If a is taken from B instead of A, then (1) defines the structure of a left B-module on $A \otimes_B V$, and there is a natural B-homomorphism

$$\epsilon : v \to 1 \otimes v, \qquad v \in V, \tag{2}$$

of V into $A \otimes_B V$, such that $A \otimes_B V$ is generated, as an A-module, by $\epsilon(V)$. In case A is the group algebra of a finite group, and B is the group algebra of a subgroup H of G, the representation of G afforded by the module $A \otimes_B V$ is the induced representation, defined first by Frobenius, of the representation of H afforded by V. The theory of *induced modules* $A \otimes_B V$ in general has been treated extensively by Higman **(3)**, **(4)**, and Hochschild **(5)**.

The purpose of this note is to investigate the following question.

(I) Let V be a left B-module. Does there exist a B-homomorphism π of $A \otimes_B V$ onto V such that $\pi\epsilon = 1$, where ϵ is given by (2)?

The existence of π is clearly equivalent to the requirement that ϵ map V monomorphically (that is, with kernel zero) onto a B-direct summand of $A \otimes_B V$.

The condition (I) is satisfied for all left B-modules V in case B is a semi-simple ring. Higman has observed in **(2)** that (I) holds for all left B-modules V whenever A is the group algebra of a finite group over an arbitrary field, and B the group algebra of a subgroup of G. The question (I) for general non-semi-simple rings B is of interest for the following reason. Following Jans **(6)**, we say that a ring A with left minimum condition has *unbounded representation type* if there exist indecomposable left A-modules with arbitrarily long composition series. Jans **(6)** and others have discovered criteria for A to be of unbounded representation type in case A is a finite dimensional

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algebra over a field. Little seems to be known, however, about the following question.

(II) Let B be a subring of A, and suppose that B has unbounded representation type. Does A then have unbounded representation type also?

In § 3 we shall give examples to show that in general the answers to both questions are negative. Our contribution to the study of (II) is the remark, already made in the group algebra case by Higman **(2)** that an affirmative answer to (I) for all left B-modules V implies an affirmative answer to (II). To prove this assertion, let V be an indecomposable left B-module which possesses a composition series. Then $A \otimes_B V$ is a finitely generated left B-module, and has a composition series both as a left B-module and as a left A-module. Then $A \otimes_B V$ is a direct sum of a finite number of indecomposable left A-modules $U_1, \ldots, U_r$. Each U_i in turn is a direct sum of indecomposable B-submodules. By (I), $A \otimes_B V$ has a B-direct summand isomorphic to V. Therefore, by the Krull–Schmidt Theorem, some U_{i_0} has a B-direct summand isomorphic to V, and hence U_{i_0} has a composition series as a B-module at least as long as a composition series for V. Because any irreducible left A-module is a homomorphic image of the finitely generated left B-module A, we see that the B-composition length of all irreducible left A-modules is bounded by the B-composition length of A. Combining our remarks, we conclude that if B has indecomposable modules with arbitrarily long composition series, the same assertion holds for A, as we wished to prove.

2. Main results. We give some sufficient conditions for condition (I) to hold. The first is rather trivial, but it includes the group algebra case. By a *projection* of A upon B we shall mean an endomorphism ϕ of the additive group of A such that $\phi(A) = B$, and $\phi(b) = b$ for all $b \in B$.

THEOREM 1. *Suppose there exists a projection τ of A upon B which is both a left and right B-homomorphism. Then condition (I) holds for every left B-module V.*

Proof. Let ϵ be the homomorphism of V into $A \otimes_B V$ given by (2). Then $\pi = \tau \otimes 1$ defines a homomorphism of $A \otimes_B V$ onto V such that $\pi\epsilon = 1$, because τ is a right B-homomorphism of A upon B which reduces to the identity on B. Because τ is also a left B-homomorphism of A onto B, π is a B-homomorphism, and the Theorem is proved.

COROLLARY. (*Higman* **(2)**.) *Let $A = KG$ be the group algebra of a finite group G over a field K, and let $B = KH$ be the group algebra of a subgroup H of G. Then (I) holds for every left B-module V.*

Proof. Let $g_1 = 1, g_2, \ldots, g_r$ be a set of representatives of the left cosets g_iH of H in G. Then A is a free right B-module with basis $g_1, \ldots, g_r$. Then the mapping $\tau: \sum g_i b_i \to b_1$, $b_i \in B$, is a projection of A upon B which satisfies the hypothesis of Theorem 1.

For the rest of the section we assume that there exists a projection λ of A upon B which is assumed only to be a left B-homomorphism. This hypothesis is automatically satisfied, for example, whenever B is a quasi-Frobenius ring, since a quasi-Frobenius ring B is an injective left (as well as right) B-module (see **(1)**). With the projection λ we shall associate a two-sided ideal I_λ in B which measures the extent to which λ fails to be a right B-homomorphism. We begin by defining for each $a \in A$ and $b \in B$ the element

$$f(a, b) = \lambda(ab) - \lambda(a)b$$

of B. We then define I_λ to be the set of all finite sums $\sum f(a_i, b_i)$, with a_i in A and b_i in B. The function f satisfies the conditions

$$f(-a, b) = f(a, -b) = -f(a, b),$$
$$bf(a_i, b_i) = f(ba_i, b_i),$$

and

$$f(a_i, b_i)b = f(a_i, b_ib) - f(a_ib_i, b),$$

for all $a, a_i \in A$ and $b, b_i \in B$. From these formulas it follows at once that I_λ is a two-sided ideal.

Our main result can be stated as follows.

THEOREM 2. *Let B be a subring of A such that A is a projective right B-module, and let λ be a left B-projection of A upon B with associated ideal I_λ in B. Then for every left B-module V such that $I_\lambda V = 0$, the mapping $\epsilon: v \to 1 \otimes v$ maps V monomorphically onto a B-direct summand of $A \otimes_B V$.*

Proof. We have to prove that there exists a B-homomorphism π of $A \otimes_B V$ onto V such that $\pi\epsilon = 1$. Because A is a projective right B-module, the first theorem of **(7)** implies that A is a direct sum of submodules a_iB, $1 \leqslant i \leqslant s$, such that each a_iB is B-isomorphic to a right ideal e_iB in B generated by an idempotent e_i. The isomorphism θ_i of a_iB onto e_iB can be chosen so that $\theta_i(a_i) = e_i$, $1 \leqslant i \leqslant s$, and it follows that $a_ie_i = a_i$ for each i. From the properties of tensor products, it follows that $A \otimes_B V$ can be expressed as a direct sum

$$A \otimes_B V = \sum_{i=1}^{s} \oplus (a_iB \otimes V) = \sum_{i=1}^{s} \oplus (a_i \otimes V). \tag{3}$$

For any element $x = \sum a_i \otimes v_i$, $v_i \in V$, of $A \otimes_B V$, we define

$$\pi(x) = \sum_{i=1}^{s} \lambda(a_i)e_iv_i. \tag{4}$$

First we check that π is well-defined. If $x = 0$, then because of the direct sum decomposition (3) we have $a_i \otimes v_i = 0$ for $i = 1, \ldots, s$. Moreover, since $\theta_i(a_ib)v = \theta_i(a_i)bv$, $b \in B$, $v \in V$, there exists a homomorphism σ_i of $a_iB \otimes V$ into V such that $\sigma_i(a_ib \otimes v) = \theta_i(a_ib)v$, $b \in B$, $v \in V$, and we have

$$0 = \sigma_i(a_i \otimes v_i) = \theta_i(a_i)v_i = e_iv_i.$$

Then from (4) we have $\pi(x) = 0$. π is obviously additive. We next verify that $\pi\epsilon = 1$. We can express the identity element 1 in A in the form $1 = \sum a_i b_i$, where we may assume that $e_i b_i = b_i$, $1 \leqslant i \leqslant s$. Then for $v \in V$, we have

$$\pi\epsilon(v) = \pi(1 \otimes v) = \pi\Big(\sum a_i \otimes b_i v\Big) = \sum \lambda(a_i) e_i b_i v = \sum \lambda(a_i) b_i v$$

$$= \sum (\lambda(a_i b_i) - f(a_i, b_i))v = \sum \lambda(a_i b_i)v = \lambda(1)v = v$$

since $\lambda(1) = 1$, and $I_\lambda V = 0$. Finally we prove that π is a B-homomorphism. Let $b \in B$; then we have for each i,

$$ba_i = \sum_{j=1}^{s} a_j \beta_{ji},$$

where the $\beta_{ji} \in B$, and we may assume that $e_j \beta_{ji} e_i = \beta_{ji}$ for all i and j. Then we have

$$b\pi\Big(\sum a_i \otimes v_i\Big) = b \sum \lambda(a_i) e_i v_i = \sum \lambda(ba_i) e_i v_i,$$

while on the other hand we have

$$\pi\Big(b \sum a_i \otimes v_i\Big) = \pi\Big(\sum_i \sum_j a_j \beta_{ji} \otimes v_i\Big) = \pi\Big(\sum_j a_j \otimes \sum_i \beta_{ji} v_i\Big)$$

$$= \sum_j \lambda(a_j) e_j \Big(\sum_i \beta_{ji} v_i\Big) = \sum_j \sum_i \lambda(a_j) \beta_{ji} e_i v_i$$

$$= \sum_j \sum_i (\lambda(a_j \beta_{ji}) - f(a_j, \beta_{ji})) e_i v_i = \sum_i \lambda\Big(\sum_j a_j \beta_{ji}\Big) e_i v_i$$

$$= \sum_i \lambda(ba_i) e_i v_i = b\pi\Big(\sum a_i \otimes v_i\Big),$$

since $f(a_j, \beta_{ji}) \in I_\lambda$ and $I_\lambda V = 0$. This completes the proof of the theorem.

From Theorem 2 and the remarks in § 1, we have the following corollary.

COROLLARY 1. *Let A and B satisfy the hypothesis of Theorem 2, and suppose that B/I_λ has unbounded representation type. Then A has unbounded representation type.*

COROLLARY 2. *Let A be a projective right B-module, where B is a quasi-Frobenius ring. Moreover, for some left B-projection λ of A upon B, let $NI_\lambda = 0$, where N is the radical of B. Then for every finitely generated left B-module V, $\epsilon: v \to 1 \otimes v$ maps V monomorphically onto a B-direct summand of $A \otimes_B V$.*

Proof. Because V is finitely generated, V is a direct sum of indecomposable left B-modules V_i, $1 \leqslant i \leqslant t$. If we can prove Corollary 2 for each V_i, then it is clear that Corollary 2 will hold for V. Therefore we may assume that V is indecomposable. The hypothesis that $NI_\lambda = 0$ implies that I_λ is a sum of minimal left ideals in B. We prove first that $I_\lambda V \neq 0$ implies that V is injective. This result is a familiar one in the theory of quasi-Frobenius rings (see **(8)**), but for the sake of completeness we sketch the proof. We have $I_\lambda = \sum I_\lambda e_i$, where the e_i are primitive idempotents in B. Because $NI_\lambda = 0$, $I_\lambda e_i \neq 0$

implies that $I_\lambda e_t$ is the unique minimal subideal in the indecomposable left ideal Be_t of B. Now $I_\lambda V \neq 0$ implies that for some e_t, $I_\lambda e_t V \neq 0$, and there exists $v \in V$ such that $I_\lambda e_t v$ is a non-zero submodule of V. The mapping $be_t \to be_t v$ is a B-homomorphism of Be_t onto $Be_t v$, and since Be_t has a unique minimal subideal not contained in the kernel of the homomorphism, it follows that $Be_t v$ is isomorphic to Be_t. On the other hand, Be_t is an injective left B-module; hence $Be_t v$ is an injective submodule of V. Since V is indecomposable we must have $Be_t v = V$, and V is injective. We have now shown that for a given indecomposable left B-module V, either $I_\lambda V = 0$ and Theorem 2 applies to V, or V is injective. It remains to prove that if V is injective, then $\epsilon: v \to 1 \otimes v$ maps V monomorphically onto a B-direct summand of $A \otimes_B V$. Because B is quasi-Frobenius, B is also injective as a right B-module, and there exists a right B-projection ρ of A upon B. Then $(\rho \otimes 1)\epsilon = 1$, and it follows that ϵ is a monomorphism of V into $A \otimes_B V$. Because $\epsilon(V)$ is injective, $\epsilon(V)$ is a B-direct summand of $A \otimes_B V$, and Corollary 2 is proved.

3. Examples. First we give an example to show that in general the answer to (II) is "no" even when both A and B are quasi-Frobenius rings. Let K be any field of characteristic $p > 0$, and let B be the group algebra over K of any finite group with a non-cyclic p-Sylow subgroup. Higman has proved in **(2)** that B has unbounded representation type. Moreover B is quasi-Frobenius (in fact a symmetric algebra), and can be imbedded in the algebra A consisting of all n by n matrices over K, where n is the dimension of B over K. But A is a simple algebra, and has only one indecomposable module. Therefore the answer to (II) is negative in this case, and there must also exist left B-modules for which (I) does not hold either.

Finally we give an example of a pair (A, B), with B quasi-Frobenius, and A a free right B-module, such that for some left B-projection λ of A onto B, we have $I_\lambda = B$. We show, furthermore, that in this case there do exist left B-modules V such that $\epsilon(V)$ is not a direct summand of $A \otimes_B V$.

Let K be an arbitrary field, and let n be an even integer, $n \geqslant 2$. Let I denote the n by n identity matrix, and J the n by n matrix $e_{21} + e_{32} + \ldots + e_{n,n-1}$, where e_{ij} denotes the matrix with a 1 in the (i, j) position and zeros elsewhere. Let A be the algebra of $2n$ by $2n$ matrices generated by the identity matrix 1, and the matrices

$$a = \begin{pmatrix} 0 & I \\ I & I \end{pmatrix}, \qquad b = \begin{pmatrix} J & I + J \\ 0 & -J \end{pmatrix},$$

where the entries in a and b stand for n by n blocks. Then we have

$$a^2 = a + 1, \qquad b^{n-1} \neq 0, \qquad b^n = 0, \qquad ab + ba = b + 1.$$

Let $B = K[b]$; then B is a quasi-Frobenius subalgebra of A, and A is a free right B-module with basis $\{1, a\}$. The elements $\{1, a\}$ form also a basis for A as a free left B-module, so that every element in A can be expressed

uniquely in the form $\beta_0 + \beta_1 a$, with β_0 and β_1 in B. Define a mapping $\lambda: A \to B$ by setting $\lambda(\beta_0 + \beta_1 a) = \beta_0$. Then λ is a left B-homomorphism of A onto B whose restriction to B is the identity mapping. The ideal I_λ defined by λ contains

$$\lambda(ab) - \lambda(a)b = \lambda(-ba + b + 1) = b + 1,$$

which is an invertible element in B since b is nilpotent. Therefore $I_\lambda = B$. Now let L be a left ideal in B; then $AL \cong A \otimes_B L$, and the mapping $\epsilon: l \to l$ is a B-homomorphism of $L \to AL$. In particular, let $L = Kb^{n-1}$; then $AL = Kb^{n-1} + Kab^{n-1}$. Let M be a left B-submodule of AL not contained in L. Then M must contain an element

$$m = \xi b^{n-1} + \eta a b^{n-1}, \qquad \xi, \eta \in K, \qquad \eta \neq 0.$$

Then

$$bm = \eta(-ab + b + 1)b^{n-1} = \eta b^{n-1} \in M,$$

and $M \cap \epsilon(L) \neq 0$. Therefore $\epsilon(L)$ is not a left B-direct summand of AL.

References

1. S. Eilenberg and T. Nakayama, *On the dimension of modules and algebras II*, Nagoya Math. J., *9* (1955), 1–16.
2. D. G. Higman, *Indecomposable representations at characteristic p*, Duke Math. J., *21* (1954), 377–381.
3. ——— *Induced and produced modules*, Can. J. Math., *7* (1955), 490–508.
4. ——— *Relative cohomology*, Can. J. Math., *9* (1957), 19–34.
5. G. Hochschild, *Relative homological algebra*, Trans. Amer. Math. Soc., *82* (1956), 246–269.
6. J. P. Jans, *On the indecomposable representations of algebras*, Ann. of Math., *66* (1957), 418–429.
7. H. Nagao and T. Nakayama, *On the structure of the (M_0) and (M_u)-modules*, Math. Zeit., *59* (1953), 164–170.
8. C. Nesbitt and R. Thrall, *Some ring theorems with applications to modular representations*, Ann. of Math., *47* (1946), 551–567.

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AN INVERSION FORMULA FOR THE WEIERSTRASS TRANSFORM

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1. Introduction. The Weierstrass transform $f(x)$ of a function $\phi(y)$ is defined by

$$f(x) = \int_{-\infty}^{+\infty} k\left(\frac{x-y}{2}\right)\phi(y)dy \tag{1.1}$$

where

$$k(x) = (4\pi)^{-\frac{1}{2}}e^{-x^2}$$

whenever this integral exists **(7**, p. 174**)**. It is also known as the Gauss transform **(11**; **12)**. Its basic properties have been developed and studied in **(7)** and in particular it has been shown that the symbolic operator

$$e^{-D^2}, \qquad D \equiv \frac{d}{dx}$$

will invert this transform under suitable assumptions and with certain definitions of this operator. We propose to study the definition

$$e^{-D^2}f(x) = \lim_{n\to\infty}\left(1 - \frac{D^2}{n}\right)^n f(x) \tag{1.2}$$

for $f(x)$ in C^∞. This formula seems to have been first examined by Pollard **(9)** and later by Rooney **(12)**. In so far as convergence of (1.2) is concerned, we will considerably improve the results of **(12)**.

2. The inversion operator. Along with Rooney **(12)**, we note that (1.2) is in reality a summability of the series arising from the following interpretation of e^{-D^2}

$$e^{-D^2}f(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}D^{2n}f(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}f^{(2n)}(x). \tag{2.1}$$

For a general series $\sum_{n=0}^{\infty} c_n$, the summability process is defined as

$$\lim_{n\to\infty}\sum_{k=0}^{n} a_{k,n}c_k = \lim_{n\to\infty}\sum_{k=0}^{n}\frac{n!k}{(n-k)!n^{k+1}}s_k$$

where

$$s_k = \sum_{i=0}^{k} c_i, \qquad a_{k,n} = \frac{n!}{(n-k)!n^k}.$$

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This process has been studied by Amerio **(1)**, Bernstein **(3)**, and Rey-Pastor **(10)**. We will refer to this as (S) summability. This follows the definition of Amerio. It is in fact regular. We will compare it with the familiar Euler (E, q) method defined as follows **(6**, p. 180**)**,

$$\lim_{n\to\infty} (q+1)^{-n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} s_k, \qquad 0 < q < \infty$$

for the series mentioned above.

THEOREM 1. *Let* $\sum_{n=0}^{\infty} c_n$ *be summable* (E, q) *for* q *satisfying*

$$(2.2) \qquad (2q+1) < \exp[(2q+1)^{-1} + 1],$$

then the series is summable (S).

Let q_0 be the unique root of the equation

$$2q_0 + 1 = \exp[(2q_0 + 1)^{-1} + 1].$$

Clearly $0 < q < q_0$. From **(6**, p. 181**)**, $c_n = o[(2q+1)^n]$, $n \to \infty$. The series $\sum_{n=0}^{\infty} c_n z^n$ then converges for $|z| < (2q+1)^{-1}$. In particular it converges for $z = Re^{i\theta}$ for any θ and $R = (2q'+1)^{-1}$, $(2q+1) < (2q'+1) < (2q_0+1)$. By a result of Bernstein **(3**, p. 358**)**, the series $\sum_{n=0}^{\infty} c_n z^n$ is summable (S) up to and including the value $z = x_0$, where x_0 is defined by $x_0/R = \exp((x_0/R)^{-1}+1)$. Clearly $(x_0/R) = 2q_0 + 1$; that is $x_0 = R(2q_0+1) = (2q_0+1)/(2q'+1) > 1$ so that $\sum_{n=0}^{\infty} c_n$ is summable (S).

The series is indeed summable to the same value. This along with a different proof of Theorem 1 has been shown in **(4**, p. 78**)**. The number q_0 is approximately 1.29 . . . . Rey-Pastor **(10)** has proven that the series

$$\sum_{n=0}^{\infty} (-1)^n a^n$$

is summable (S) to $(1+a)^{-1}$ for $-1 < a < a_0$, $a_0 = 2q_0 + 1$, $a_0 = 3.59 \ldots$. The process diverges for a outside this region. On the other hand, (E, q) sums this series to $(1+a)^{-1}$ for $-1 < a < 2q+1$ and thus we see that Theorem 1 is best possible.

Returning to the definition (1.2), we obtain

$$(2\,3) \qquad \left(1 - \frac{D^2}{n}\right)^n f(x) = \int_{-\infty}^{+\infty} K_n\left(\frac{x-y}{2}\right) k\left(\frac{x-y}{2}\right) \phi(y)\,dy$$

where

$$(2{,}4) \qquad K_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (4n)^{-k} H_{2k}(x)$$

and $H_n(x)$ is the Hermite polynomial defined by

$$H_n(x) = (-1)^n e^{x^2} D^n e^{-x^2}.$$

We begin with

LEMMA 1.

(a) $$\int_{-\infty}^{+\infty} k\left(\frac{x}{2}\right)K_n\left(\frac{x}{2}\right)dx = 1.$$

(b) $$K_n(x) = \frac{(-n)^n\sqrt{n}\,n!}{(2n)!}\int_0^\infty e^{-nt}t^{-\frac{1}{2}}(1+t)^n H_{2n}\left(\frac{x\sqrt{t}}{(1+t)^{\frac{1}{2}}}\right)dt.$$

Part (a) follows from the orthogonality of Hermite polynomials. On the other hand, from Bailey **(2)**,

$$u^{2n}H_{2n}\left(\frac{x}{u}\right) = \sum_{k=0}^{n}\frac{(2n)!}{k!(2n-2k)!}(1-u^2)^k H_{2n-2k}(x).$$

Let $u^2 = (t+1)/t$, multiply both sides by e^{-nt} and integrate from 0 to ∞ to get

$$\int_0^\infty e^{-nt}t^{-\frac{1}{2}}(1+t)^n H_{2n}\left(\frac{x\sqrt{t}}{(1+t)^{\frac{1}{2}}}\right)dt$$
$$= (-1)^n(2n)!\sum_{k=0}^{n}(-1)^k\frac{\Gamma(k+\frac{1}{2})}{(n-k)!(2k)!}n^{-k-\frac{1}{2}}H_{2k}(x)$$

and this is part (b).

Some bounds for $K_n(x)$ are now developed.

LEMMA 2. (a) $K_n(x) = O(n^{\frac{1}{2}})$, $n\to\infty$, *uniformly for x in any finite interval.*

(b) $K_n(x) = O[n^{\frac{1}{2}}\exp\{(x^2/2) + n - n(1+2x^2/n)^{\frac{1}{2}} + n\log(\frac{1}{2}+\frac{1}{2}(1+(2x^2/n)^{\frac{1}{2}})\}]$ *uniformly for $0 \leqslant x \leqslant A\sqrt{n}$, A any constant.*

(c) $K_n(x) = O(n^{-\frac{1}{2}}x^{-2}) + O(n^{-\gamma})$ *where γ is some positive number and the result holds uniformly for $0 < x \leqslant n^{\epsilon}$, $\epsilon < 1/24$.*

(d) $|K_n(x)| < (1+2x^2/n)^n$, $|x| > (2n+\frac{1}{2})^{\frac{1}{2}}$.

It is a consequence of a result proved in **(13**, p. 194**)** that $H_{2n}(x) = O[((2n)!/n!)]$ uniformly for x in any finite interval. Thus by Lemma 1,

$$K_n(x) = O(n^{\frac{1}{2}})\int_0^\infty e^{-nt}t^{-\frac{1}{2}}(1+t)^n dt, \qquad n\to\infty.$$

An appeal to a result in **(5**, p. 37**)** finishes the proof of part (a). For part (b) we use the result, **(13**, p. 236**)**,

$$e^{-x^2/2}H_{2n}(x) = O[2^n\{(2n)!\}^{\frac{1}{2}}]$$

uniformly for all x. Thus

$$K_n(x) = O(n^{\frac{1}{2}})\int_0^\infty e^{h_n(t)}\frac{dt}{\sqrt{t}(1+t)}$$

where $h_n(t) = -nt + (n+1)\log(1+t) + (x^2t/2(1+t))$. $h_n(t)$ has a maximum at $t = t_0$,

$$t_0 = \tfrac{1}{2}\left[-1+\left(1+\frac{2x^2}{n}\right)^{\frac{1}{2}}\right] + O\left(\frac{1}{n}\right), \qquad n\to\infty$$

uniformly for $0 \leqslant x < \infty$. Then $K_n(x) = O[n^{\frac{1}{3}}e^{h_n(t_0)}]$ and it is easy to show that

$$h_n(t_0) = n - n\left(1 + \frac{2x^2}{n}\right)^{\frac{1}{2}} + n \log\left[\tfrac{1}{2} + \tfrac{1}{2}\left(1 + \frac{2x^2}{n}\right)^{\frac{1}{2}}\right] + \frac{x^2}{2} + O(1)$$

uniformly for $0 \leqslant x \leqslant A\sqrt{n}$ for any constant A. This is part (b). For the more difficult part (c), we use the following asymptotic formula for $H_n(x)$ **(13**, p. 195**)**,

$$e^{-u^2/2}H_n(u) = 2^{n/2+\frac{1}{4}}(n!)^{\frac{1}{2}}(\pi n)^{-\frac{1}{4}}(\sin\phi)^{-\frac{1}{2}} \left\{\sin\left[\left(\frac{n}{2} + \frac{1}{4}\right)(\sin 2\phi - 2\phi) + \frac{3\pi}{4}\right] + O(n^{-1})\right\}$$

for $u = (2n+1)^{\frac{1}{2}}\cos\phi$ and the order condition holds uniformly for $\mu \leqslant \phi \leqslant \pi - \mu$, $\mu > 0$. Thus, after using Stirling's formula, we get

$$K_n(x) = \frac{(-1)^n\sqrt{n}}{\sqrt{\pi}}\int_0^\infty e^{-nt}t^{-\frac{1}{2}}(1+t)^n e^{x^2t/2(1+t)}(\sin\phi)^{-\frac{1}{2}} \sin\left[(n+\tfrac{1}{4})(\sin 2\phi - 2\phi) + \frac{3\pi}{4}\right]dt + O(n^{-\frac{1}{2}})\int_0^\infty e^{-nt}t^{-\frac{1}{2}}(1+t)^n e^{x^2t/2(1+t)}dt$$

with

$$x\left(\frac{t}{1+t}\right)^{\frac{1}{2}} = (4n+1)^{\frac{1}{2}}\cos\phi.$$

The second part is $O(n^{-\frac{1}{2}}e^{h_n(t_0)})$ with $h_n(t_0)$ as before and we will now assume that $0 \leqslant x \leqslant n^\epsilon$ for ϵ as yet unspecified but less than $\frac{1}{2}$. We split the remaining integral into two parts I_1 and I_2 corresponding respectively to the ranges $(0, t_0 + n^{\delta-\frac{1}{2}})$, $(t_0 + n^{\delta-\frac{1}{2}}, \infty)$, t_0 as above and $0 < \delta < \frac{1}{2}$. $h_n(t)$ is a decreasing function in the range defining I_2 so that

$$I_2 = O(e^{h_n(t_0+n^{\delta-\frac{1}{2}})})$$

and we can show that $h_n(t_0 + n^{\delta-\frac{1}{2}}) \leqslant h_n(t_0) - cn^{2\delta}$ for some constant c and $0 < \delta < 1/6$. Thus $I_2 = O(e^{h_n(t_0)-cn^{2\delta}})$ and

$$K_n(x) = (-1)^n\left(\frac{n}{\pi}\right)^{\frac{1}{2}}I_1 + O(n^{-\frac{1}{2}}e^{h_n(t_0)}), \qquad n \to \infty$$

$0 \leqslant x \leqslant n^\epsilon$, $0 < \epsilon < \frac{1}{2}$. Now we can also show for the same range of x, $h_n(t_0) = O(n^{4\epsilon-1}) + O(1)$, $n \to \infty$ so that for $0 < \epsilon < \frac{1}{4}$,

$$K_n(x) = (-1)^n\left(\frac{n}{\pi}\right)^{\frac{1}{2}}I_1 + O(n^{-\frac{1}{2}}), \qquad n \to \infty$$

where

$$I_1 = \int_0^{t_0+n^{\delta-\frac{1}{2}}} t^{-\frac{1}{2}}e^{g_n(t)}(\sin\phi)^{-\frac{1}{2}}\sin\left[(n+\tfrac{1}{4})(\sin 2\phi - 2\phi) + \frac{3\pi}{4}\right]dt$$

with

$$x\left(\frac{t}{1+t}\right)^{\frac{1}{2}} = (4n+1)^{\frac{1}{2}}\cos\phi, \quad g_n(t) = -nt + n\log(1+t) + \frac{x^2 t}{2(1+t)}.$$

For $0 < \epsilon < \delta/2 < 1/24$, $g_n(t) = (-nt^2/2) + O(n^{-\frac{1}{2}-\gamma})$ uniformly for $0 \leqslant x \leqslant n^{\epsilon}$ where γ is some small positive number. Also

$$(\sin\phi)^{-\frac{1}{2}} = 1 + O(n^{-\frac{1}{2}-\gamma}),$$

$$\sin\left[(n+\tfrac{1}{4})(\sin 2\phi - 2\phi) + \frac{3\pi}{4}\right]$$
$$= (-1)^n \cos[(4n+1)^{\frac{1}{2}} x\sqrt{t} - \tfrac{1}{2}(4n+1)^{\frac{1}{2}} x t^{3/2}] + O(n^{3\delta-7/4})$$

and the last term is $O(n^{-\frac{1}{2}-\gamma})$ for $\delta < 1/12$. Then we can show that

$$K_n(x) = \frac{n^{1/4}}{\sqrt{\pi}} \int_0^{\sqrt{n}\,t_0+n^{\delta}} e^{-u^2/2} u^{-\frac{1}{2}}$$
$$\cos\left[\frac{(4n+1)^{\frac{1}{2}}}{n^{\frac{1}{2}}} x\sqrt{u} - \frac{(4n+1)^{\frac{1}{2}}}{2n^{\frac{1}{2}}} x u^{3/2}\right] du + O(n^{-\gamma})$$

uniformly for $0 \leqslant x \leqslant n^{\epsilon}$. The upper limit may be replaced by ∞ and the error is

$$O(n^{\frac{1}{2}} e^{-cn^{2\delta}}).$$

We now split up the integral according to the addition formula for the cosine function into J_1 and J_2. J_1 corresponds to the integral with the cosines in the integrand. Two integrations by parts will show that $J_1 = O(n^{-\frac{1}{2}}x^{-2}) + O(n^{-\frac{1}{2}-\gamma})$. In a similar way, the second part $J_2 = O(n^{-\frac{1}{2}})$. Combining all of these results, we obtain

$$K_n(x) = O(n^{-\frac{1}{2}}x^{-2}) + O(n^{-\gamma}), \qquad n \to \infty$$

uniformly for $0 < x \leqslant n^{\epsilon}$.

For part (d), we use the bound for $H_n(x)$ developed in **(8**, p. 158**)**,

$$|H_n(x)| < 2^{3n/2}\,|x|^n, \qquad |x| > (n+\tfrac{1}{2})^{\frac{1}{2}}$$

so that for $|x| > (2n+\frac{1}{2})^{\frac{1}{2}}$

$$|K_n(x)| < \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{4n}\right)^k 2^{3k} x^{2k} = \left(1 + \frac{2x^2}{n}\right)^n.$$

This completes the proof of the lemma.

3. The main theorem. We begin with a lemma on the roots of a certain transcendental equation.

LEMMA 3. *There is one and only one solution to the pair of equations*

$$1 + (A - \tfrac{1}{2})\frac{k^2}{4} - \left(1 + \frac{k^2}{2}\right)^{\frac{1}{2}} + \log\left[\tfrac{1}{2} + \tfrac{1}{2}\left(1 + \frac{k^2}{2}\right)^{\frac{1}{2}}\right] = 0$$

$$-(1 - A)\frac{k^2}{4} + \log\left(1 + \frac{k^2}{2}\right) = 0$$

outside of the trivial pair $k^2 = 0$, $A =$ *anything.*

The change of variable $u = (1 + k^2/2)^{\frac{1}{2}}$ and the elimination of A results in

$$\frac{(u-3)(u-1)}{4} = \log\left(\frac{2u^2}{1+u}\right)$$

and it is easy to see from this that there are only two solutions and that $u = 1$ corresponds to the trivial solution. The important solution (k_0^2, A_0) is

$$A_0 = .738\ldots, \qquad k_0^2 = 49.876\ldots .$$

We now prove our main result.

THEOREM 2. *Suppose* $A < A_0$, *where* A_0 *is defined in Lemma* 3 *and that*

$$\int_{-\infty}^{+\infty} e^{-A(y^2 - 2|y|)/4} |\phi(y)| dy < \infty.$$

If, moreover, $\phi(x_0+)$ *and* $\phi(x_0-)$ *exist and* $f(x)$ *is the Weierstrass transform of* ϕ, *then*

$$\lim_{n\to\infty}\left(1 - \frac{D^2}{n}\right)^n f(x)\Big]_{x=x_0} = \tfrac{1}{2}[\phi(x_0+) + \phi(x_0-)].$$

Let $\Lambda = \frac{1}{2}[\phi(x_0+) + \phi(x_0-)]$. By Lemma 1, we can write

$$\left(1 - \frac{D^2}{n}\right)^n f(x) - \Lambda = \int_{-\infty}^{+\infty} k\left(\frac{x_0 - y}{2}\right) K_n\left(\frac{x_0 - y}{2}\right)[\phi(y) - \Lambda] dy.$$

The integral is decomposed into I_1, I_2, I_3 corresponding respectively to the ranges $(-\infty, x_0 - \eta)$, $(x_0 - \eta, x_0 + \eta)$, $(x_0 + \eta, \infty)$, $\eta > 0$. Again decompose I_3 into I_3', I_3'', I_3''' corresponding respectively to $(x_0 + \eta, x_0 + k_0 n^{\epsilon})$, $(x_0 + k_0 n^{\epsilon}, x_0 + k_0\sqrt{n})$, $(x_0 + k_0\sqrt{n}, \infty)$ with ϵ, k_0 as before. Now, by Lemma 2,

$$|I_3'''| \leqslant \int_{k_0\sqrt{n}}^{\infty} e^{g(u)} e^{-Au^2/4} |\phi(u + x_0) - \Lambda| du$$

where $g(u) = -(1 - A)u^2/4 + n\log(1 + (u^2/2n))$. $g(u)$ has a maximum for

$$u = u_0 = \sqrt{n}\left(2\frac{(1+A)}{(1-A)}\right)^{\frac{1}{2}}.$$

Then

$$u_0 < \sqrt{n}\left(2\frac{(1+A_0)}{(1-A_0)}\right)^{\frac{1}{2}} < k_0\sqrt{n}.$$

Also $g'(u) < 0$ in the range of integration so that

$$I_3''' = O[e^{g(k_0\sqrt{n})}] \int_{k_0\sqrt{n}}^{\infty} e^{-Au^2/4} |\phi(u + x_0) - \Lambda| du.$$

But

$$g(k_0\sqrt{n}) = n\left[- (1 - A)\frac{k_0^2}{4} + \log\left(1 + \frac{k_0^2}{2}\right)\right]$$

$$< n\left[- (1 - A_0)\frac{k_0^2}{4} + \log\left(1 + \frac{k_0^2}{2}\right)\right] = 0$$

and thus $I_3''' = o(1)$, $n \to \infty$. On the other hand, by Lemma 2

$$I_3'' = O(n^{\frac{1}{4}}) \int_{k_0 n^\epsilon}^{k_0\sqrt{n}} e^{d(u)} e^{-Au^2/4} |\phi(u + x_0) - \Lambda| du.$$

Now

$$(3.1) \qquad d(u) = n + (A - \tfrac{1}{2})\frac{u^2}{4} - n\left(1 + \frac{u^2}{2n}\right)^{\frac{1}{2}} + n \log\left[\tfrac{1}{2} + \tfrac{1}{2}\left(1 + \frac{u^2}{2n}\right)^{\frac{1}{2}}\right]$$

$$= (A - A_0)\frac{u^2}{4} + h(u).$$

By hypothesis $h(k_0\sqrt{n}) = 0$ and also $h(0) = 0$. It is not difficult to show that $h(u) \leqslant 0$ for $0 \leqslant u \leqslant k_0\sqrt{n}$. Then from (3.1)

$$\max d(u) \leqslant \tfrac{1}{4}(A - A_0)(k_0 n^\epsilon)^2.$$

Therefore

$$I_3'' = O(n^{\frac{1}{4}} e^{(k_0^2/4)(A - A_0)n^{2\epsilon}}) \int_{k_0 n^\epsilon}^{k_0\sqrt{n}} e^{-Au^2/4} |\phi(u + x_0) - \Lambda| du$$

and this is $o(1)$. Now by Lemma 2,

$$I_3' = O(n^{-\frac{1}{4}}) \int_{\eta}^{k_0 n^\epsilon} e^{-u^2/4} u^{-2} |\phi(u + x_0) - \Lambda| du$$

$$+ O(n^{-7}) \int_{\eta}^{k_0 n^\epsilon} e^{-u^2/4} \phi(u + x_0) - \Lambda| du$$

and this is also $o(1)$, $n \to \infty$. Thus $I_3 = o(1)$. The proof that $I_1 = o(1)$ is similar. There remains I_2. Since $K_n(x)$ is even,

$$(3.2) \qquad I_2 = \int_{x_0}^{x_0+\eta} k\left(\frac{x_0 - y}{2}\right) K_n\left(\frac{x_0 - y}{2}\right) [\phi(y) - \phi(x_0+)] dy$$

$$+ \int_{x_0-\eta}^{x_0} k\left(\frac{x_0 - y}{2}\right) K_n\left(\frac{x_0 - y}{2}\right) [\phi(y) - \phi(x_0 -)] dy.$$

For the first integral, given $\delta > 0$ choose η such that $|\phi(y) - \phi(x_0+)| < \delta$ for $x_0 \leqslant y \leqslant x_0 + \eta$. Then

$$\int_{x_0}^{x_0+\eta} k\left(\frac{x_0-y}{2}\right)\left|K_n\left(\frac{x_0-y}{2}\right)\right| \, |\phi(y)-\phi(x_0+)|dy < \delta \int_0^{\eta} e^{-u^2/4}\left|K_n\left(\frac{u}{2}\right)\right| du$$

and this last integral is split up according to the ranges $(0, n^{-\frac{1}{2}})$, $(n^{-\frac{1}{2}}, \eta)$. By Lemma 2,

$$\int_0^{n^{-\frac{1}{2}}} e^{-u^2/4}|K_n\left(\frac{u}{2}\right)|du = O(n^{\frac{1}{2}}) \int_0^{n^{-\frac{1}{2}}} e^{-u^2/4}du = O(1), \qquad n \to \infty.$$

Also

$$\int_{n^{-\frac{1}{2}}}^{\eta} e^{-u^2/4}|K_n\left(\frac{u}{2}\right)|du = O(n^{-\frac{1}{2}}) \int_{n^{-\frac{1}{2}}}^{\eta} e^{-u^2/4}u^{-2}du + O(n^{-\gamma}) \int_{n^{-\frac{1}{2}}}^{\eta} e^{-u^2/4}du$$

$$= O(n^{-\frac{1}{2}}) \int_{n^{-\frac{1}{2}}}^{\eta} u^{-2}du + O(n^{-\gamma}) = O(1).$$

Thus the first integral in (3.2) is $\delta O(1), n \to \infty$. The same holds true for the second integral and therefore

$$\overline{\lim} \, |I_2| < M\delta, \; M \text{ a constant.}$$

This proves the theorem.

This result should be compared with the theorem on convergence of (2.1). The series inverts approximately when $A = \frac{1}{2}$ (referring to the A of Theorem 2). See **(11)** and **(4**, p. 12**)**. Not all functions which are Weierstrass transforms can be inverted by our inversion formula. For example **(7**, p. 178**)**, let

$$f(x) = e^{ax^2} = \int_{-\infty}^{+\infty} k\left(\frac{x-y}{2}\right)(1+4a)^{-\frac{1}{2}}e^{(a/1+4a)y^2}dy$$

for $-\frac{1}{4} < a < \infty$. The series (2.1) is now

$$e^{ax^2}\sum_{n=0}^{\infty}\frac{a^n}{n!}H_{2n}(\sqrt{-a}\,x)$$

which for $x = 0$ is

$$\sum_{n=0}^{\infty}(-1)^n\frac{(2n)!}{(n!)^2}a^n.$$

This is $(1+4a)^{-\frac{1}{2}}$ for $|4a| < 1$. It can be shown **(4**, p. 107**)** that this series is summable (S) to $(1+4a)^{-\frac{1}{2}}$ for $(4a) < \exp((1/4a)+1)$ and diverges otherwise. That is, it diverges for $a > .897\ldots$.

References

1. L. Amerio, *Un metodo di sommazione per le serie di potenze*, Ann. Scuola Norm. Sup. Pisa, Sci. Fis. Mat., *2*, 8 (1939), 167–180.
2. W. N. Bailey, *An integral representation for the product of two Hermite polynomials*, J. London Math. Soc., *13* (1948), 202–203.

3. S. Bernstein, *Sur le convergence de certaines suites des polynomes*, J. Math. Pures et Appl., *9*, 15 (1936), 345–358.
4. G. G. Bilodeau, *Inversion and representation theory for the Weierstrass transform* (Thesis, Harvard, March 1959).
5. A. Erdelyi, *Asymptotic expansions* (Dover, 1956).
6. G. H. Hardy, *Divergent series* (Oxford, 1949).
7. I. I. Hirschman and D. V. Widder, *The convolution transform* (Princeton, N.J., 1955).
8. E. Kogbetliantz, *Recherches sur la sommabilité des séries d'Hermite*, Ann. Sci. de L'Ecole Norm. Sup., *49*, 3 (1932), 147–221.
9. H. Pollard, *Integral transforms*, Duke Math. J., *13* (1946), 307–330.
10. J. Rey-Pastor, *Un metodo de sumacion de series*, Rend. Circ. Mat. Palermo, *55* (1931), 450–455.
11. P. G. Rooney, *On the inversion of the Gauss transformation*, Can. J. Math., *9* (1957), 459–464.
12. ——— *On the inversion of the Gauss transformation II*, Can. J. Math., *10* (1958), 613–616.
13. G. Szegö, *Orthogonal polynomials* (New York, 1939).

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TRANSFORMATION ALGEBRAS

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Introduction. The purpose of this paper is to show that most results concerning polyadic algebras can be generalized to transformation algebras. The results of this paper will clearly indicate that a great deal can be done in polyadic algebras without ever mentioning the quantifier structure (for instance, terms and operations can be characterized without the help of the quantifier structure, at least in the case where an equality is present). In § 1, we develop the elementary theory; in § 2, we study the different ways of extending a (locally finite) transformation algebra (of infinite degree) to a polyadic algebra; in § 3, we study equality transformation algebras; finally, in § 4, we show how terms and operations can be defined in equality transformation algebras.

0. Preliminaries. The purpose of this section is to establish the notation and the basic terminology to be used throughout this paper. For easier reference, we also state, without proof, one known lemma. The notation for Boolean algebras and polyadic algebras is the one used by Halmos in his four papers **(1–4)**. However, there are two notational changes which we describe now. If I is a set and i, j are elements of I, then the symbol (j/i) denotes the transformation on I which sends i onto j and every other element onto themselves. Previously, in **(3)**, this transformation was denoted by (i/j). The advantage of this new notation is that the "cancellation law" holds: $S(j/k)S(k/i)p = S(j/i)p$ whenever p belongs to some polyadic algebra and i, j, and k are variables so that p is independent of k. Accordingly, if t is a term and K a set of variables, then the symbol (K/t) as defined in **(3)** shall be replaced by (t/K). The second change of notation concerns functional polyadic algebras. Suppose X and I are sets, J a subset of I, and x, y belong to X^I. We define $x \equiv y \bmod J$ (to read: x is equal to y modulo J) to mean $x_i = y_i$ whenever i is not in J. The letter O shall denote the two-element Boolean algebra. A homomorphism from a Boolean algebra B onto O shall be called a *valuation* of B. A *sum* of B is a subset Σ of B so that Σ has a supremum in B which is denoted by $\vee \Sigma$. If C is a Boolean algebra and f is a mapping from B into C, then f is said to *preserve* a sum Σ of B whenever $f(\Sigma)$ is a sum of C and $f(\vee \Sigma) = \vee f(\Sigma)$. If f is a homomorphism from B into C and if f preserves every sum of B, then f is called a *complete homomorphism*. Suppose C is a subalgebra of B and Σ is a sum of C. Then B is said to *preserve* the sum Σ if the inclusion mapping of C into B preserves the sum Σ.

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Suppose B is a Boolean algebra and V is a set of valuations of B. The smallest topology on B which makes every valuation in V continuous (the topology on O is the discrete one) will be referred to as the *V-topology* or the *topology induced by V*. With the V-topology B becomes a bounded topological Boolean algebra, that is, B viewed as a group under $+$ is a bounded topological group; moreover, the Boolean operations are uniformly continuous. If the V-topology is Hausdorff, the topological completion $\bar{B}$ of B is a compact Hausdorff topological Boolean algebra, and $\bar{B}$ is isomorphic and homeomorphic to the Boolean algebra of all subsets of V, the topology on $\bar{B}$ being induced by the set of all complete valuations of $\bar{B}$. A set V of valuations of B is said to be *separating (on B)* if for every p in B distinct from 0, there is a valuation v in V so that $v(p) = 1$. The V-topology is Hausdorff if and only if V is separating. All the results just stated are easy to establish, and, for that reason, we omit the proofs. For more details concerning bounded topological Boolean algebras, see **(6)**. We conclude this section by stating a known result.

(0.1) LEMMA. **(9)** *If B is a Boolean algebra, then there exists a unique complete Boolean algebra A so that B is a subalgebra of A, A preserves every sum of B, and for every element p in A, there exists a subset Σ of B, so that $p = \vee \Sigma$. Moreover, if C is a complete Boolean algebra and f is a complete homomorphism from B into C, then f has a unique extension to a complete homomorphism from A into C.*

1. Elementary theory. We begin with the definition of a transformation algebra; it was first formulated by Halmos. A *transformation algebra* is a triple (B, I, S) where B is a Boolean algebra, I a set, and S a mapping from transformations τ on I to endomorphisms $S(\tau)$ of B so that

(S_1) $S(\delta)$ is the identity endomorphism if δ is the identity on I;

(S_2) $S(\tau\sigma) = S(\tau)S(\sigma)$ whenever τ and σ are transformations on I. As in the polyadic case, we shall often commit the solecism of identifying (B, I, S) with B and we shall also say that B is an I-algebra. It is clear that, if $(B, I, S, \exists)$ is a polyadic algebra, then (B, I, S) is a transformation algebra. For that reason, we shall simply say that a polyadic algebra is a transformation algebra. However, in general, the converse is not true. The elementary algebraic theory for transformation algebras is straightforward and very similar to that of polyadic algebras. In the sequel we shall use expressions like transformation subalgebras, transformation homomorphisms, transformation ideals, and the like without explicit reference. Their meaning is clear. The *degree* of (B, I, S) is the cardinality of I. The concept of independence is formulated as follows: if $p \in B$ and J is a subset of I, then p is said to be *independent* of J, if, for all transformations τ on I with $\tau i = i$ for i in $I - J$, $S(\tau)p = p$. A subset K of I is said to be a *support* for p if p is independent of $I - K$. It is true, as in the polyadic case, that if $p \in B$, K is a support for p and τ is a transformation on I, then $\tau(K)$ supports $S(\tau)p$. It is easy to see that if B happens to be a polyadic algebra, then the concept of support as defined above coincides with

the concept of support as defined in **(2)**. If every element in B has finite support, then B is said to be *locally finite*. If C is a Boolean algebra and X a set, then a C-valued functional transformation algebra over X is defined as in the polyadic case, except that no mention is made of quantification. We shall also make use of this concept without explicit reference.

(1.1) THEOREM. *Every transformation algebra is isomorphic to a functional transformation algebra.*

Proof. Let A be the transformation algebra of all functions from I^I into B and define a mapping f from B into A by $(fp)(\tau) = S(\tau)p$ (remember that a transformation on I is an element of I^I and conversely). It is easy to check that f is actually a transformation isomorphism of B into A.

(1.2) COROLLARY. *Any transformation algebra can be embedded in a polyadic algebra.*

Proof. Because of (1.1), it suffices to prove the corollary for a functional transformation algebra. If B is a C-valued functional transformation algebra over a set X, let $\bar{C}$ be a complete Boolean algebra containing C and let A be the polyadic algebra of all functions from X^I into $\bar{C}$. Then A is clearly a polyadic extension of B.

Suppose B is a transformation I-algebra. If i is a variable, p an element of B, then the set of all elements of B of the form $S(j/i)p$ for some j in I shall be called an *existential subset* of B, and it shall be denoted by $\Sigma(i, p)$; if $\Sigma(i, p)$ has a supremum in B, it shall be called an *existential sum* of B. Note that what is called an existential subset here was called a totally existential subset in **(5)**. A valuation v of B will be called an *existential valuation* if it preserves every existential sum of B. The relevance of the concept of existential subset is shown by the following results.

(1.3) LEMMA. *Suppose $(A, I, S, \exists)$ is a locally finite polyadic algebra of infinite degree. If I_0 is an infinite subset of I, i a variable in I, and p an element of A, then*

$$\exists(i)p = \vee\{S(j/i)p : j \in I_0\}.$$

Proof. If i is in I_0, fix the variables in $I - I_0$, and apply **(2**, 10.5**)**. If i is not in I_0, choose k in I_0 so that p is independent of k. Since $S(j/k)S(k/i)p = S(j/i)p$ for all j in I_0, it follows that $\vee\{S(j/i)p : j \in I_0\} = \exists(k)S(k/i)p = \exists(i)p$. This completes the proof of the Lemma.

It follows from (1.3) that if $\Sigma(i, p)$ is an existential subset of A, then $\Sigma(i, p)$ is a sum and $\exists(i)p$ is the supremum of $\Sigma(i, p)$. It follows also that a transformation isomorphism from A onto another polyadic algebra is necessarily a polyadic isomorphism. In particular, this implies that if a transformation algebra can be made into a polyadic algebra, then this can be done in only one way.

(1.4) LEMMA. *If τ is a transformation on I so that $\tau(I)$ is infinite, then $S(\tau)$ preserves the existential sums of A; if $\tau(I) = I$, and if Σ is an existential sum of A, then $S(\tau)(\Sigma)$ is an existential sum of A.*

Proof. Let $\tau(I) = I_0$ and let $\Sigma(i, p)$ be an existential sum of A. Suppose σ is a finite transformation on I so that σ agree with τ on a finite support J of p, and let k be a variable so that $\sigma^{-1}k = \{k\}$ and k is not in $\tau(J)$. Then $S(\tau)\exists(i)p = S(\sigma)\exists(i)p = \exists(k)S(\sigma)S(k/i)p = \vee\{S(j/k)S(\sigma)S(k/i)p : j \in I_0\} = \vee\{S(\tau)S(j/i)p : j \in I\}$. Since $\exists(i)p$ is the supremum of $\Sigma(i, p)$, it follows that $S(\tau)$ preserves $\Sigma(i, p)$. This proves the first part of the Lemma. The second part is immediate.

The following theorem is more or less the converse of (1.4).

(1.5) THEOREM. *Suppose (B, I, S) is a locally finite transformation algebra of infinite degree. If every existential subset of B is a sum and if $S(\tau)$ preserves the existential sums whenever $\tau(I) = I$, then there is a unique mapping $\exists$ from subsets of I to quantifiers of B so that $(B, I, S, \exists)$ is a polyadic algebra.*

Proof. Let A be the transformation algebra of all finite dimensional functions from I^I into B. Define a mapping f from B into A by $(fp)(\tau) = S(\tau)p$ for every transformation τ on I and every p in B. Let $\bar{B} = f(B)$. Clearly f is a transformation isomorphism. Since uniqueness of the quantifier mapping $\exists$ is an immediate consequence of (1.3), to complete the proof it suffices to show that $\bar{B}$ is a functional polyadic algebra. For that purpose, let p in B, i in I, $q = \vee\,\Sigma(i, p)$, and σ and τ transformations on I so that $\tau(I) = I$ and $\sigma \equiv \tau \bmod (I - J)$ where J is a finite support for both p and q, and $i \in J$. Then $(fq)(\sigma) = S(\sigma)q = S(\tau)q = \vee\{S(\tau)S(j/i)p : j \in I\} = \vee\{S(\pi)p : \pi \equiv \tau \bmod i\} = \vee\{(fp)(\pi) : \pi \equiv \sigma \bmod i\} = (\exists(i)fp)(\sigma)$ which shows that $\exists(i)$ is defined on $\bar{B}$ and therefore, since $\bar{B}$ is locally finite $\exists(J)$ is defined for every subset J of I. It follows that $\bar{B}$ is a functional, polyadic algebra. This completes the proof of the theorem.

We close this section by observing that if B has an equality (see § 3), then the hypothesis, in the preceding theorem, that $S(\tau)$ preserves the existential sums whenever $\tau(I) = I$ can be removed. The results of § 3 will clearly indicate that this hypothesis is necessarily satisfied in the presence of an equality.

2. Polyadic completions of a transformation algebra. The purpose of this section is to study the different ways of extending a transformation algebra to a polyadic algebra. We begin with two Lemmas.

(2.1) LEMMA. *Suppose A is a locally finite polyadic I-algebra of infinite degree. If A has cardinality less than or equal to that of I, and if V is the set of all existential valuations of A, then V is separating on A.*

Proof. Because of semi-simplicity, it suffices to prove the lemma in the case where A is simple. In view of the cardinality assumption and of the representation theorem for polyadic algebras, we may in turn assume that A is an O-valued algebra over X where X has cardinality less than or equal to that of I. Choose q in A so that $q \neq 0$. Choose x in X^I so that $x(I) = X$ and so that $q(x) = 1$. Define a valuation v of A by $vp = p(x)$ for all p in A; clearly $vq = 1$. It is a straightforward matter to check that v is an existential valuation of A. This shows that V is separating on A.

We make the observation that (2.1) is false if the cardinality assumption is removed. Indeed, let A be the algebra of all finite dimensional functions from X^I into O where I is countably infinite and X has cardinality greater than that of the continuum. Suppose v is an existential valuation of A. Let $\bar{A}$ be the algebra of all finite dimensional functions from I^I into O and define a mapping f from A into $\bar{A}$ by $fp(\tau) = vS(\tau)p$. Then f is an isomorphism into $\bar{A}$ which is a contradiction since A and $f(A)$ have different cardinalities. This shows that A has no existential valuation.

(2.2) LEMMA. *Suppose A is a locally finite polyadic algebra of infinite degree and B is a transformation subalgebra which generates A polyadically. If V is a set of existential valuations of A, and if A is topologized with the V-topology, then B is dense in A.*

Proof. Clearly, we may assume that V contains all existential valuations of A. In view of (1.4), it follows that $S(\tau)$ is continuous whenever τ is a transformation on I so that $\tau(I) = I$. Denote by $\bar{B}$ the closure of B; then $\bar{B}$ is a transformation subalgebra. To see that $\bar{B} = A$, it suffices to prove that $\bar{B}$ is a polyadic subalgebra. For that purpose, let p in $\bar{B}$ and i in I. For each finite subset J, let $p_J = \vee \{S(j/i)p : j \in J\}$. Because every valuation in V preserves the existential sums, it follows that the net $\{p_J : J \in D\}$ converges to $\exists(i)p$ where D is the directed set of all finite subsets of I, the ordering on D being set-theoretical inclusion. Hence $\exists(i)p$ belongs to $\bar{B}$ which shows that $\bar{B} = A$.

If B is a transformation algebra, then a *polyadic completion* of B is a pair (A, h) where A is a polyadic algebra and h a transformation homomorphism from B into A such that $h(B)$ generates A; (A, h) shall be called *faithful* if h is a transformation monomorphism. When no confusion is liable to arise, A itself shall be called a polyadic completion of B. We shall give now, under suitable hypotheses, a complete classification of all possible polyadic completions of a given locally finite transformation algebra of infinite degree.

For the remainder of this section, (B, I, S) shall be a fixed locally finite transformation algebra of infinite degree.

Suppose (A_1, h_1) and (A_2, h_2) are polyadic completions of B. Then a *homomorphism* from (A_1, h_1) onto (A_2, h_2) is a polyadic homomorphism f from A_1 onto A_2 such that $fh_1 = h_2$; if f is an isomorphism, then (A_1, h_1) and (A_2, h_2)

are said to be *isomorphic*. We are now ready to prove the main theorem of this section concerning the existence of polyadic completions.

(2.3) THEOREM. *Suppose V is a non-empty set of valuations of B. Then there exists a unique (up to isomorphism) polyadic completion (A, h) of B having the following two properties:*

(2.4) *for every v in V, there is a (necessarily unique) existential valuation $\bar{v}$ of A so that $\bar{v}h = v$;*

(2.5) *for every p in A distinct from 0, there is a valuation v in V and a transformation τ on I so that $\bar{v}S(\tau)p = 1$.*

(A, h) shall be called the *polyadic completion (of B) with respect to V* or else the *V-polyadic completion (of B)*.

Proof. Let $\bar{V}$ be the set of all valuations of B of the form $vS(\tau)$ where v is in V and τ is a transformation on I so that $\tau(I) = I$. Assume first that $\bar{V}$ is separating on B. Topologize B with the $\bar{V}$-topology and let $\bar{B}$ be the topological completion of B; then $\bar{B}$ is actually isomorphic and homeomorphic with the Boolean algebra of all subsets of $\bar{V}$, the topology on $\bar{B}$ being induced by the set of all complete valuations. If τ is a transformation on I so that $\tau(I) = I$, then $S(\tau)$ is uniformly continuous on B and therefore $S(\tau)$ has a unique extension to a complete endomorphism $\bar{S}(\tau)$ of $\bar{B}$. Let A^* be the set of all p in $\bar{B}$ for which there is a finite subset J of I so that $S(\tau)p = S(\sigma)p$ whenever $\tau(I) = \sigma(I) = I$ and $\tau \equiv \sigma \bmod (I - J)$; then A^* is a Boolean algebra. It is easy to see that there is a unique way of extending $\bar{S}$ to all transformations on I so that $(A^*, I, \bar{S})$ becomes a locally finite transformation algebra. Moreover, if p is in A^*, i in I, and $q = \vee \{\bar{S}(j/i)p : j \in I\}$ where the supremum is taken in $\bar{B}$, then q is in A^*. This follows from the fact that $\bar{S}(\tau)$ is a complete endomorphism of $\bar{B}$ whenever $\tau(I) = I$. It follows from (1.5) that there is a unique quantifier mapping $\exists$ so that $(A^*, I, \bar{S}, \exists)$ is a polyadic algebra. Let A be the polyadic subalgebra generated by B; if h is the identity on B, then (A, h) is a polyadic completion of B having the properties (2.4) and (2.5). In the general case, that is, when $\bar{V}$ is not necessarily separating, let M be the set-theoretical intersection of all ideals of the form $v^{-1}(0)$ where $v \in \bar{V}$. Let $\bar{B} = B/M$ and let h be the natural projection from B onto $\bar{B}$. Since M is a transformation ideal, h is a transformation homomorphism. Let W be the set of all valuations of $\bar{B}$ so that $v \in W$ if and only if there is $\bar{v}$ in $\bar{V}$ such that $\bar{v}p = vhp$ for every p in B. If τ is a transformation on I so that $\tau(I) = \mathrm{I}$, and if v is in W, then $vS(\tau)$ is in W. Clearly W is separating on $\bar{B}$. We are then reduced to the first case which concludes the proof as far as existence is concerned. Uniqueness follows from (2.2) together with the fact that if (A, h) is a polyadic completion satisfying (2.4) and (2.5), then $h^{-1}(0) = M$.

(2.6) THEOREM. *If B has cardinality less than or equal to that of I, then any*

polyadic completion of B is the polyadic completion of B with respect to some set of valuations.

Proof. Suppose (A, h) is a polyadic completion of B. Let $\bar{V}$ be the set of all existential valuations of A. Let V be the set of all valuations of B of the form vh where $v \in \bar{V}$. Since B has cardinality less than or equal to that of I, A has also the same property. By (2.1), (A, h) satisfies (2.4) and (2.5). By uniqueness, (A, h) is the polyadic completion of B with respect to V.

The following theorem describes the main properties of the polyadic completions of B.

(2.7) THEOREM. *If V_1 and V_2 are sets of valuations of B so that V_2 is a subset of V_1, then the V_2-polyadic completion is a quotient of the V_1-polyadic completion; if (A, h) is the V-polyadic completion, then (A, h) is faithful if and only if $\bar{V}$ is separating where $\bar{V}$ is the set of all valuations of the form $vS(\tau)$ with v in V and τ a transformation on I so that $\tau(I) = I$; if B has cardinality less than or equal to that of I, then a polyadic completion of B is simple if and only if it is the polyadic completion with respect to a single valuation.*

Proof. The proof of the first part follows easily from a continuity argument. The proof of the second part consists in a straightforward verification. To prove the third part, suppose (A, h) is the polyadic completion of B with respect to a single valuation. It follows from (2.5) that A cannot contain closed elements distinct from 0 and 1. Therefore, A is simple. Conversely, suppose A is simple. Since A has also a cardinality less than or equal to that of I, by (2.1), we can assert that there is at least one existential valuation $\bar{v}$ of A. Let v be a valuation of B defined by $v = \bar{v}h$. Then (A, h) satisfies (2.4) and (2.5) with respect to v. By uniqueness, it follows that (A, h) is the polyadic completion with respect to v. This completes the proof of the theorem.

3. Equality transformation algebras. If B is a locally finite transformation algebra of infinite degree, then a *predicate* of B is defined as in **(3)**. An *equality* for B is a reflexive and substitutive binary predicate of B. An equality E for B is necessarily symmetric and transitive. Moreover, $p \wedge E(i, j) = S(j/i)p \wedge E(i, j) = S(i, j)p \wedge E(i, j)$ whenever i and j are variables and $p \in B$. The proof of these elementary facts goes exactly as in the polyadic case (treated in **(4)**, § 2) without any change.

For the remainder of this section, (B, I, S) shall be a fixed locally finite transformation algebra of infinite degree with an equality E.

Suppose (A, h) is a polyadic completion of B and suppose moreover that Σ is a sum of B; (A, h) is said to *preserve* Σ if h preserves Σ. The main purpose of this section is to show that there exists one and only one faithful polyadic completion of B which preserves every sum of B. From this result follows an interesting refinement of (1.1) (see 3.4). We first prove two Lemmas.

(3.1) LEMMA. *If (A, h) is a polyadic completion of B, then hE is an equality for A.*

Proof. What is meant here is that, if $F(i, j) = hE(i, j)$ for all variables i and j, then F is an equality for A. Clearly, we may assume that A is faithful and that h is the identity endomorphism of B onto B. Let A_0 be the set of all p in A so that $p \wedge E(i, j) = S(j/i)p \wedge E(i, j)$ whenever i and j are variables. Since A_0 is a polyadic subalgebra of A and since A_0 contains B, it follows that $A_0 = A$. This completes the proof of the Lemma.

(3.2) LEMMA. *If τ is a finite transformation on I, then $S(\tau)$ is a complete endomorphism of B.*

Proof. If τ is a transposition, then $S(\tau)$ is an automorphism of B and hence $S(\tau)$ preserves the sums of B. Therefore, since every finite transformation is a finite product of transpositions and replacements, it suffices to prove the Lemma for the case, where $\tau = (j/i)$ and $j \neq i$. Let Σ be a sum of B and let $p_0 = \vee \Sigma$. Suppose q is an element in B so that $S(j/i)p \leqslant q$ for all p in Σ. We are to show that $S(j/i)p_0 \leqslant q$. For that purpose, let k be a variable distinct from i and j and so that p and q are independent of k. Since $S(j/i)p$ is independent of i for all p, we have $S(j/i)p \leqslant S(k/i)q$ for all p in Σ. But $p \wedge E(i, j) \leqslant S(j/i)p$ for all p in Σ which implies that $p \wedge E(i, j) \leqslant S(k/i)q$ for all p in Σ. It follows that $p_0 \wedge E(i, j) \leqslant S(k/i)q$. Applying $S(j/i)$ to both sides of the last inequality, we get $S(j/i)p_0 \leqslant S(k/i)q$. It follows that $S(i/k)S(j/i)p_0 \leqslant S(i/k)S(k/i)q = q$. But since $S(i/k)S(j/i)p_0 = S(j/i)p_0$, it follows finally that $S(j/i)p_0 \leqslant q$. This shows that $S(j/i)$ preserves the sum Σ.

We make the observation that the preceding lemma is false for polyadic algebras with no equality and *a fortiori* for transformation algebras with no equality. Counterexamples can be furnished. We are now ready to prove the main result of this section.

(3.3) THEOREM. *There exists a unique (up to isomorphism) faithful polyadic completion of B which preserves every sum of B.*

Proof. Let $\bar{B}$ be the McNeille completion of B (see 0.1), and let $\bar{A}$ be the polyadic algebra of all finite dimensional functions from I^I into $\bar{B}$. Define a mapping h from B into $\bar{A}$ by $(hp)(\tau) = S(\tau)p$ for all p in B and all τ in I^I. The mapping h is a transformation isomorphism from B into $\bar{A}$. Let A be the polyadic subalgebra of $\bar{A}$ generated by $h(B)$. Clearly, (A, h) is a faithful polyadic completion of B. Using (3.2), and using the fact that an element of $\bar{A}$ is completely determined by its values on finite transformations, it is easy to see that (A, h) preserves the sums of B. This proves existence. Uniqueness follows from (0.1) and (3.2).

The polyadic completion (A, h) given by the preceding theorem shall be referred to as the *McNeille polyadic completion of B*.

We conclude this section by giving an application of (3.3). First, we need two definitions. An element p in B is said to be *undecidable* if $p \neq 0$ and if there exists an element q in B so that $q \neq 1$ and $S(\tau)p \leqslant q$ for all transformations τ on I. It follows from **(2**, 10.3**)** that, if B is a polyadic algebra, then p is undecidable if and only if $\exists(I)p \neq 0$ and $\exists(I)p \neq 1$. Suppose A is a locally finite functional transformation algebra of functions from X^I into a Boolean algebra C. The functional transformation algebra A is said to be an *existential functional (transformation) algebra* if, whenever an existential sum $\Sigma(i, p)$ of A has a supremum q in A, then $q(x)$ is the supremum (in C) of $\{p(y) : x \equiv y \bmod i\}$ for all x in X^I. Clearly, if A is a functional polyadic algebra, then A is necessarily an existential functional algebra. The following theorem is a direct generalization of the representation theorem for locally finite polyadic algebras of infinite degree.

(3.4) THEOREM. *The transformation algebra B is isomorphic to an existential functional transformation algebra; B is isomorphic to an O-valued existential functional transformation algebra if and only if B has no undecidable elements.*

Proof. The first part of the theorem follows immediately from **(2**, 10.9**)** and (3.3). To prove the second part, in view of **(2**, 17.3**)**, it suffices to prove that the McNeille polyadic completion A of B is simple if and only if B has no undecidable element. This follows easily from **(2**, 10.3**)** and (0.1).

4. Terms and operations in transformation algebras. With the results so far obtained, it is now possible to develop the theory of terms and operations in transformation algebras, at least in the case where an equality is present. The purpose of this section is to show how most definitions and results of **(3)** can be generalized to locally finite equality transformation algebras of infinite degree.

Throughout this section (B, I, S) shall be a fixed locally finite transformation algebra of infinite degree with an equality E.

We begin by stating the definition of a J-constant of B where J is a finite subset of I. A *J-constant* of B is a mapping c that associates with each subset K of $I - J$ an endomorphism $S(c/K)$ of B so that:

(T_1) if ϕ is the empty subset of I, then $S(c/\phi)$ is the identity on B;

(T_2) $S(c/H \cup K) = S(c/H)S(c/K)$ whenever H and K are subsets of $I - J$;

(T_3) $J \cup (L - K)$ supports $S(c/K)p$ whenever K is a subset of $I - J$ and L supports p;

(T_4) $S(c/K)p = S(c/K \cap L)p$ whenever K is a subset of $I - J$ and L is a support for p;

(T_5) $S(c/K)S(\tau) = S(\tau)S(c/\tau^{-1}K)$ whenever K is a subset of $I - J$ and τ is a transformation that lives on $I - J$.

A *constant* of B is a ϕ-constant where ϕ is the empty set. If B is a polyadic algebra, then a J-constant of B as defined in **(3)** is a J-constant in the preceding sense; this follows from **(3**, (5.1) and (5.2)**)**. The results of this section will

imply that the converse is also true (see (4.2)). Now that we have the definition of a J-constant, the definition of a J-term goes exactly as in the polyadic case. Namely, a *J-term* of B is a mapping t that associates with every subset K of I a Boolean endomorphism $S(t/K)$ of B, so that the restriction of t to subsets of $I - J$ is a J-constant of B, and so that $S(t/K)p = S(t/\sigma K)S(\sigma)p$ whenever p is an element of B with finite support L, K is a finite subset of I and σ is a transformation of type $(K, L \cup J)$.

If c is a J-constant and i a variable, we define $E(i, c)$ by $E(i, c) = S(c/k)E(i, k)$ where $k \neq i$ and k is in $I - J$. Using T_3 and T_5, it is easy to see that $E(i, c)$ is unambiguously defined, that is, independent of the choice of k. We define $E(c, i)$ in a similar manner and clearly $E(i, c) = E(c, i)$. Note that $S(c/i)E(i, c) = 1$ and $S(j/i)E(i, c) = E(j, c)$ whenever i and j are variables not in J. We are now ready to prove the main result concerning J-constants.

(4.1) THEOREM. *Suppose c is a J-constant and suppose i is a variable not in J. If $q = E(i, c)$, then, for all p in B, $\Sigma(i, p \wedge q)$ is an existential sum and*

$$S(c/i)p = \vee \Sigma(i, p \wedge q).$$

Proof. Let p be a fixed element of B. We first show that if j is any variable, then $S(j/i)(p \wedge q) \leqslant S(c/i)p$. Assume first j distinct from i and not in J. The required inequality follows then by applying $S(c/i)$ to both sides of the inequality $S(j/i)p \wedge E(i, j) \leqslant p$. In the general case, choose a variable k distinct from i, not in J, and so that p, q, and $S(c/i)p$ are independent of k. The required inequality follows then by applying $S(j/k)$ to both sides of the inequality $S(k/i)(p \wedge q) \leqslant S(c/i)p$. To complete the proof, it suffices to show that if p_0 is an element of B so that $S(j/i)(p \wedge q) \leqslant p_0$ for all variables j, then $S(c/i)p \leqslant p_0$. To prove that, select a variable j not in J and so that p and p_0 are independent of j. If we apply $S(c/j)$ to both sides of the inequality $S(j/i)p \wedge E(j, c) \leqslant p_0$, we get $S(c/j)S(j/i)p \leqslant S(c/j)p_0 = p_0$. Since $S(c/j)S(j/i)p = S(c/i)p$, we have finally $S(c/i)p \leqslant p_0$. This completes the proof of the theorem.

(4.2) COROLLARY. *If B is a polyadic algebra, then a J-constant as defined above is a J-constant as defined in* **(3)**.

Proof. This follows from the preceding theorem together with (1.3) and from the fact that if c is a J-constant (as defined in **(3)**), then $S(c/i)p = \exists(i)(p \wedge E(i, c))$ for all p in B and all variables i not in J.

Our next task is to establish the exact relationship between J-constants of B and J-constants of the polyadic completions of B. If (A, h) is a polyadic completion of B and c is a J-constant of B, then A will be said to *preserve* c if there exists a (necessarily unique) J-constant $\bar{c}$ of A so that $hS(c/K)p = S(\bar{c}/K)hp$ whenever K is a subset of $I - J$ and p belongs to B.

(4.3) THEOREM. *Suppose c is a J-constant of B, i a variable not in J, $q = E(i, c)$, and (A, h) a polyadic completion of B. Then A preserves c if and only if A preserves the existential sum $\Sigma(i, q)$.*

Proof. First note that, by (4.1), the supremum (in B) of $\Sigma(i, q)$ is 1. Assume first that A preserves c. Let $\bar{c}$ be the unique J-constant of A so that $hS(c/K)p = S(\bar{c}/K)hp$ for all p in B and all subsets K of $I - J$. By (3.1), hE is an equality for A; let $hE = \bar{E}$ and let $\bar{q} = \bar{E}(i, c)$. By (4.1), the supremum (in A) of $\Sigma(i, \bar{q})$ is 1. Since $hS(j/i)q = S(j/i)\bar{q}$ for all j in I, we conclude that h (and therefore A) preserves $\Sigma(i, q)$. Conversely, suppose A preserves the sum $\Sigma(i, q)$. Let $\bar{q} = hE(i, c)$. It is easy to verify that $\exists!(i)\bar{q} = 1$. By **(4,** (9.1)**)**, there exists a unique J-constant $\bar{c}$ of A so that $S(\bar{c}/i)\bar{q} = 1$. Since $S(\bar{c}/i)hp = \exists(i)(hp \wedge \bar{q})$ for all p in B, it follows from (1.3) and (4.1) that $S(\bar{c}/i)hp \leqslant hS(c/i)p$ for all p in B which implies that $S(\bar{c}/i)hp = hS(c/i)p$ for all p in B. Therefore A preserves c. This completes the proof of the theorem.

If (A, h) is the polyadic completion with respect to a set V of valuations of B, it can be shown that A preserves c if and only if every valuation in $\hat{V}$ preserves the sum $\Sigma(i, q)$ where $\hat{V}$ is the set of all valuations of the form $vS(\tau)$ with $v \in V$ and with $\tau(I) = I$.

With these theorems at our disposal the theory of J-constants in B can now be reduced to the theory of J-constants in polyadic algebras. Indeed, let A be the McNeille polyadic completion of B; we may assume that B is a subalgebra of A. Since A preserves the sums of B, it follows by (4.3) that a J-constant of B has a unique extension to a J-constant of A. Therefore, if c is a J-constant of B, by **(3,** (6.7)**)**, there exists a unique J-term t of A so that $S(c/K)p = S(t/K)p$ whenever $p \in B$ and K is a subset of $I - J$. By **(3,** (4.1)**)**, $S(t/K)$ sends B into B for all subsets K of I; it follows that a J-constant of B has a unique extension to a J-term of B. If t is a term of B and τ is a transformation on I, then the transform τt of t by τ is a term of A (here, we identify t with its unique extension to a term of A), and by **(3,** (11.6)**)** τt is also a term of B (or more precisely, the restriction of τt to B is a term of B). Similarly, if s and t are terms of B and K is a subset of I, then the result $(s/K)t$ of replacing the variables of K in t by s is a term of B (see **(3,** (13.2)**)**). An *operation* of B is then defined as it is defined by **(3,** (12.3)**)**. We conclude by making the remark that all the definitions and results in **(3)** concerning terms and operations in polyadic algebras are valid in (locally finite) equality transformation algebras (of infinite degree) as long as no essential use of quantification is made.

Errata. The results of this paper were announced in **(6)** and **(7)**; the results announced in **(5)** combined with those of the present paper lead to some applications given in **(8)**. The purpose of this section is to correct an error that appeared in two different forms in **(6)** and **(7)**. The first correction concerns Theorem VI of **(6)**: add to the statement of that theorem the hypothesis that A has cardinality less than or equal to that of I. The second correction concerns Theorem II of **(7)**: add to the statement of that theorem the hypothesis that B has cardinality less than or equal to that of I.

References

1. P. R. Halmos, *Algebraic logic* I, *Monadic Boolean Algebras*, Composito Mathematica, *12* (1955), 217–249.
2. ——— *Algebraic logic* II, *Homogeneous locally finite polyadic Boolean algebras of infinite degree*, Fund. Math., *43* (1957), 255–325.
3. ——— *Algebraic logic* III, *Predicates, terms, and operations in polyadic algebras*, Trans. Amer. Math. Soc., *83* (1956), 430–470.
4. ——— *Algebraic logic* IV, *Equality in polyadic algebras*, Trans. Amer. Math. Soc., *86* (1957) 1–27.
5. L. LeBlanc, *Dualité pour les égalités Booléennes*, Comptes rendus des séances de l'académie des sciences de Paris (May 30, 1960).
6. ——— *Les algèbres Booléennes topologiques bornées*, Comptes rendus des séances de l'académie des sciences de Paris (June 8, 1960).
7. ——— *Les algèbres de transformations*, Comptes rendus des séances de l'académie des sciences de Paris (June 13, 1960).
8. ——— *Représentation des algèbres polyadiques pour anneau*, Comptes rendus des séances de l'académie des sciences de Paris (June 20, 1960).
9. H. M. McNeille, *Partially ordered sets*, Trans. Amer. Math. Soc., *42* (1937).

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ANTICOMMUTING LINEAR TRANSFORMATIONS

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1. It is well known that any set of four anticommuting involutions (see §2) in a four-dimensional vector space can be represented by the Dirac matrices

$$(1)\quad B_{2,0} = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}, B_{2,1} = \begin{pmatrix} 0 & B_{1,0} \\ B_{1,0} & 0 \end{pmatrix}, B_{2,2} = \begin{pmatrix} 0 & B_{1,1} \\ B_{1,1} & 0 \end{pmatrix}, B_{2,3} = \begin{pmatrix} 0 & B_{1,2} \\ B_{1,2} & 0 \end{pmatrix}$$

where the $B_{1,r}$ are the Pauli matrices

$$(2)\qquad B_{1,0} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; B_{1,1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; B_{1,2} = i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

(See **(1)** for a general exposition with applications to Quantum Mechanics.) One formulation, which we shall call the Dirac-Pauli theorem **(2; 3; 1)**, is

THEOREM 1. *If M_1, M_2, M_3, M_4 are 4×4 matrices satisfying*

$$M_rM_s + M_sM_r = 2\delta_{rs}1_4 \qquad (r, s = 1, 2, 3, 4),$$

then there is a matrix T such that

$$T^{-1}M_rT = B_{2,r-1} \qquad (1 < r < 4),$$

and T is unique apart from an arbitrary numerical multiplier.

Various proofs of this theorem are known; those due to Van der Waerden **(4)** and Pauli **(2; 1)** depend on ideas belonging to representation theory; the most elementary proof (ignoring the uniqueness of T) is given by Dirac **(2)**.

Eddington has shown **(5)** that a set of anticommuting 4×4 involution matrices cannot include more than five members, and this was extended by Newman **(6)** to involution matrices of arbitrary order. This had been investigated earlier by Hurwitz **(7)**.

In this note we give a completely elementary proof of Theorem 1 (on the lines of Dirac's proof), giving an explicit calculation of T (Theorem 5 and corollary); the generalization of Theorem 1 to linear transformations of spaces of dimension 2^k is given in Theorem 7. In Theorem 2 we prove a generalization of the Eddington-Newman result in which the restriction to involution matrices is removed.

2. Notation. If V is an n-dimensional vector space, we write $d(V) = n$, and if L is a linear transformation (L.T.) of V into itself we write $d(L) = n$. If M is an $n \times n$ matrix, we write $d(M) = n$; the transpose of M is denoted

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by M'. The identity mapping in V is denoted by 1 or 1_n, and the same symbols are used to denote the unit matrix.

If λ is an eigenvalue of L, $\mathfrak{E}_\lambda(L)$ denotes the space spanned by the eigenvectors of L belonging to λ. L is called *regular* if 0 is not one of its eigenvalues. A subset S of V is said to be *stable* for L if $L(S) \subset S$. If $d(V) = n$, any ordered set $\{\beta_1, \beta_2, \ldots, \beta_n\}$ which span V is called a *basis* of V; if $\mathfrak{B}$ denotes this basis and c is a non-zero complex number then $c\mathfrak{B}$ denotes the basis $\{c\beta_1, c\beta_2, \ldots, c\beta_n\}$. We say the matrix M represents the linear transformation L in $\mathfrak{B}$ if

$$M = (m_{rs}) \text{ where } L\beta_s = \sum_{r=1}^{n} m_{rs}\beta_r \qquad (1 \leqslant s \leqslant n);$$

this is denoted by $L \sim M$ or by $L \sim M$ (in $\mathfrak{B}$) if the basis is to be made explicit. Plainly if $L \sim M$ (in $\mathfrak{B}$) then $L \sim M$ (in $c\mathfrak{B}$).

L is called an *involution* if $L^2 = 1$ and $L \neq \pm 1$; the involution matrix diag.$(1_n, -1_n)$ is denoted by I_{2n}. L_1 and L_2 are said to *anticommute* if $L_1L_2 = -L_2L_1$.

3. It will be convenient to list some elementary properties of matrices and L.T.'s: it is assumed throughout that the spaces are of finite dimensions; most of the proofs are omitted.

(i) If $d(M) = 2n$ then M anticommutes with I_{2n} if and only if

$$M = \begin{pmatrix} 0 & Q \\ P & 0 \end{pmatrix} \text{ with } d(P) = d(Q) = n,$$

and M is then an involution if and only if $PQ = 1_n$, that is,

$$M = \begin{pmatrix} 0 & P^{-1} \\ P & 0 \end{pmatrix};$$

$$\begin{pmatrix} 0 & X^{-1} \\ X & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & Y^{-1} \\ Y & 0 \end{pmatrix}$$

anticommute if and only if $X^{-1}Y = -Y^{-1}X$. If $L_1, L_2, \ldots, L_{2q}$ are anticommuting involutions then

$$i^q L_1 L_2 \ldots L_{2q}$$

is an involution which anticommutes with each of $L_1, L_2, \ldots, L_{2q}$; (the product of an *odd* set, three or more, will *commute* with the factors).

(ii) If A and B are regular and anticommute then, since

$$\det(AB) = \det(BA)\ (-1)^{d(A)},$$

and since these determinants are not zero, $d(A)$ must be even.

(iii) If L is an involution in V then for every x in V

$$(1 + L)x \in \mathfrak{E}_1(L), \quad (1 - L)x \in \mathfrak{E}_{-1}(L), \quad 2x = (1 + L)x + (1 - L)x,$$

and so V is the direct sum of $\mathfrak{E}_1(L)$ and $\mathfrak{E}_{-1}(L)$; in any basis formed by uniting a basis of $\mathfrak{E}_1(L)$ and a basis of $\mathfrak{E}_{-1}(L)$, $L \sim \text{diag.}(1_m, -1_n)$ for some m, n.

(iv) The basic simple result, to be used repeatedly, is that if S and T are L.T.'s of V, and $ST = kTS$ where k is a non-zero number, and λ is an eigenvalue of T, then S maps $\mathfrak{E}_\lambda(T)$ into $\mathfrak{E}_{\lambda/k}(T)$, and

$$S\{\mathfrak{E}_\lambda(T)\} = \mathfrak{E}_{\lambda/k}(T) \tag{3}$$

if S is regular. (The proof is trivial: $Tx = \lambda x$ implies $T(Sx) = k^{-1}S(Tx) = k^{-1}\lambda Sx$, and if S is regular we have (since $S(V) = V$ with $d(V) < \infty$) $S^{-1}T = k^{-1}TS^{-1}$, so that S^{-1} maps $\mathfrak{E}_{\lambda/k}(T)$ into $\mathfrak{E}_\lambda(T)$.)

In particular, if an involution L anticommutes with a regular S then

$$S\{\mathfrak{E}_1(L)\} = \mathfrak{E}_{-1}(L) \quad \text{and} \quad S\{\mathfrak{E}_{-1}(L)\} = \mathfrak{E}_1(L),$$

and it follows from (iii) that $d\{\mathfrak{E}_1(L)\} = d\{\mathfrak{E}_{-1}(L)\} = \frac{1}{2}d(L)$, and that, in a suitable basis of V, $L \sim I_{2n}$ $(n = \frac{1}{2}d(L))$.

(v) If S_1 and S_2 both anticommute with T, then S_1S_2 commutes with T, and so (by (iv)) every eigenspace of T is stable for S_1S_2. In particular, if S_1, S_2, S_3 are anticommuting involutions then

(4) *iS_2S_3 is an involution which maps $\mathfrak{E}_r(S_1)$ onto itself* $(r = \pm 1)$,

(this depends on (i) and (3)).

4. The following generalizes the Eddington-Newman result.

THEOREM 2. *Suppose $L_1, L_2, \ldots, L_{2k}$ are regular anticommuting L.T.'s of V; then 2^k is a divisor of $d(V)$.*

Proof. For $k = 1$ we appeal to § 3 (ii). Now suppose, if possible, that the theorem is true for $k = 1, \ldots, K-1$ but false for $k = K$. This means that there is a space W in which there are $2K$ regular anticommuting L.T.'s whereas 2^K is not a divisor of $d(W)$. We prove that this leads to a contradiction.

Let Δ be the least value which $d(W)$ can have in the conditions postulated, and suppose W chosen so that $d(W) = \Delta$. Let $\mathfrak{E}_\lambda$ be an eigenspace of L_1; by § 3(v), $\mathfrak{E}_\lambda$ is stable for L_2L_s $(3 \leqslant s \leqslant 2K)$, and the L_2L_s anticommute in $\mathfrak{E}_\lambda$; hence, by the induction hypothesis, $2^{K-1} | d(\mathfrak{E}_\lambda)$. Since $L_r(\mathfrak{E}_\lambda) = \mathfrak{E}_{\pm\lambda}$ for $1 \leqslant r \leqslant 2K$ by (3), the direct sum of $\mathfrak{E}_\lambda$ and $\mathfrak{E}_{-\lambda}$ is stable for all these L_r; thus, in a suitable basis of W,

$$L_r \sim \begin{pmatrix} M_r & P_r \\ 0 & A_r \end{pmatrix} \text{ where } d(M_r) = 2d(\mathfrak{E}_\lambda), \tag{5}$$

which means $d(M_r)$ is divisible by 2^K while $d(L_r)$ is not. Hence $d(A_r)$ is positive and not divisible by 2^K. But, by (5) and the assumption on the L_r, the A_r are regular and they anticommute; thus $d(A_r) \geqslant \Delta = d(L_r)$, which contradicts $d(M_r) > 0$.

It is shown by Newman **(6)** that if $d(V) = 2^n$ then there is a set of $2n + 1$

anticommuting involutions in V (see also § 7 below), but it must not be concluded that an arbitrary set of anticommuting involutions in V which has fewer than $2n + 1$ members is part of a maximal set. Thus, with $n = 2$,

$$B_{2,0}, \quad B_{2,1}, \quad iB_{2,0}B_{2,1}$$

anticommute, but it is easily verified (using § 3(i)) that an involution which anticommutes with the first cannot anticommute with the other two. The same is true if the L.T.'s are regular but not involutory; it is easily verified (using § 3(i)) for the anticommuting matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & -4 & 0 \end{pmatrix}$$

that if M anticommutes with the first then

$$M = \begin{pmatrix} 0 & b & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & c \\ 0 & 0 & d & 0 \end{pmatrix},$$

and for this to anticommute with the other two, M must vanish.

5. By § 3(iii) every involution in two dimensions has 1 as a simple eigenvalue.

THEOREM 3. *Let σ_1 and σ_2 be anticommuting involutions in two dimensions and β_1 the eigenvector (unique apart from a constant of multiplication) of σ_1 belonging to eigenvalue 1. Then, apart from an arbitrary numerical multiplier, $\{\beta_1, \sigma_2(\beta_1)\}$ is the only basis in which*

$$\sigma_1 \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \sigma_2 \sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{6}$$

Proof. Any basis in which (6) holds must have β_1 (or a numerical multiple of it) for its first member, and for σ_2 to have the matrix assigned in (6) the second member of the basis must be $\sigma_2(\beta_1)$. Conversely, if β_2 is defined as $\sigma_2(\beta_1)$, then $\beta_2 \in \mathfrak{E}_{-1}(\sigma_1)$ by § 3(iv), and $\sigma_2(\beta_2) = \sigma_2^2(\beta_1) = \beta_1$; hence (6) is valid in the basis $\{\beta_1, \beta_2\}$.

THEOREM 4. *Suppose σ_1 and σ_2 are anticommuting involutions in a two-dimensional space; then, the only regular* L.T.*'s which anticommute with σ_1 and σ_2 are the numerical multiples of $\sigma_1\sigma_2$ and of these the only involutions are $\pm i\sigma_1\sigma_2$.*

Proof. By Theorem 3, we may choose a basis so that (6) holds. A matrix M which anticommutes with the first in (6) has the form

$$\begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix}$$

by § 3(i), and this anticommutes with the second if and only if $p = -q$, that is,

$$M = p\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \sim p\sigma_1\sigma_2;$$

M is involutory if and only if $p^2 = -1$.

It is a consequence of Theorem 3 and § 3(iv) that if σ_1 and σ_2 are anticommuting involutions in a space V with $d(V) = 2n$ then V is the direct sum of n two-dimensional spaces each stable for σ_1 and σ_2; in a suitable basis of V

$$\sigma_r \sim \text{diag.}(B_{1,r-1}, B_{1,r-1}, \ldots, B_{1,r-1}) \qquad (r = 1, 2). \tag{8}$$

This follows from § 3(iv) whereby if $\{\beta_1, \beta_2, \ldots, \beta_{2n-1}\}$ is a basis of $\mathfrak{E}_1(\sigma_1)$ and $\beta_{2r} = \sigma_2(\beta_{2r-1})$, then β_{2r-1} and β_{2r} span a space stable for σ_1 and σ_2, and (8) will hold in the basis $\{\beta_1, \beta_2, \ldots, \beta_{2n-1}, \beta_{2n}\}$.

We now prove the Dirac-Pauli theorem (this is generalized in § 7).

THEOREM 5. *Let L_0, L_1, L_2, L_3 be anticommuting involutions in the four-dimensional space V; then there is a unique basis (apart from a numerical multiplier) such that $L_r \sim B_{2,r}$ $(r = 0, 1, 2, 3)$, the $B_{2,r}$ being defined by* (1).

Proof. Since

$$B_{2,0} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}, \qquad B_{2,1} = \begin{pmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\\ 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\end{pmatrix},$$

$$B_{2,2} = \begin{pmatrix}0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}, \qquad B_{2,3} = i\begin{pmatrix}0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\\ 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\end{pmatrix},$$

we have

$$iB_{2,2}B_{2,3} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix} \text{ and } iB_{2,3}B_{2,1} = \begin{pmatrix}0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix},$$

and hence the basic vectors $e_r = (\delta_{r1}, \delta_{r2}, \delta_{r3}, \delta_{r4})$ are completely characterized in terms of the $B_{2,r}$ as follows:

(a) e_1 is the non-zero vector (unique apart from a numerical multiplier), which is common to $\mathfrak{E}_1(B_{2,0})$ and $\mathfrak{E}_1(iB_{2,2}B_{2,3})$,

(b) $e_2 = iB_{2,3}B_{2,1}e_1$, $e_3 = B_{2,1}e_1$, $e_4 = -B_{2,1}e_2 = iB_{2,3}e_1$.

Thus, it is enough to show that (apart from a numerical multiplier)

(a′) there is just one non-zero x in $\mathfrak{E}_1(L_0)$ which satisfies $iL_2L_3x = x$, and that if β_1 is such an x, and we define

(b′) $\beta_2 = iL_3L_1\beta_1,\ \beta_3 = L_1\beta_1,\quad \beta_4 = iL_2\beta_1 = -L_1\beta_2,$

then $\{\beta_1, \beta_2, \beta_3, \beta_4\}$ is a basis of V in which $L_r \sim B_{2,r}$ $(0 \leqslant r \leqslant 3)$. Now by § 3(v) $\mathfrak{E}_1(L_0)$ is stable for iL_2L_3 and for iL_3L_1, and these involutions anticommute in $\mathfrak{E}_1(L_0)$. Hence, by Theorem 3, (a′) is true, and if $\beta_2 = iL_3L_1\beta_1$, then

$$iL_2L_3(\beta_1, \beta_2) = (\beta_1, -\beta_2) \qquad \text{and} \qquad iL_3L_1(\beta_1, \beta_2) = (\beta_3, \beta_1);$$

furthermore, $\{\beta_1, \beta_2\}$ span $\mathfrak{E}_1(L_0)$, and so β_3, β_4 may be defined as in (b′), and by § 3(iv) $\{\beta_3, \beta_4\}$ span $\mathfrak{E}_{-1}(L_0)$. Thus, in the basis $\{\beta_1, \beta_2, \beta_3, \beta_4\}$, $L_0 \sim I_2$ and by § 3(i)

$$L_r \sim \begin{pmatrix} 0 & X_r \\ X_r^{-1} & 0 \end{pmatrix} \qquad (1 \leqslant r \leqslant 3).$$

It is therefore enough to verify that $X_r = B_{1,r}$ $(1 \leqslant r \leqslant 3)$. For $r = 1, 2$ this follows from (b′) which implies also that $L_3(\beta_1, \beta_2) = (-i\beta_4, iL_1\beta_1)$ $= (-i\beta_4, i\beta_3)$; this completes the proof.

COROLLARY 1. *Theorem* 1 *follows from Theorem* 5. *If the L_r are matrices M_r (that is, V is the space of number quadruples), then $\beta_1, \beta_2, \beta_3, \beta_4$ are the columns, in order, of a matrix T which satisfies $M_rT = TB_{2,r}$ $(0 \leqslant r \leqslant 3)$. These columns are found explicitly from* (a′) *and* (b′) *viz.*

$$(M_0 - 1_4)\beta_1 = (M_2M_3 + i1_4)\beta_1 = 0,$$

and

$$\beta_2 = iM_3M_1\beta_1, \qquad \beta_3 = M_1\beta_1, \qquad \beta_4 = iM_2\beta_1.$$

COROLLARY 2. *If J_r $(0 \leqslant r \leqslant 3)$ are anticommuting involutions in V (of Theorem* 5*) then there is a regular* L.T., $\mathfrak{T}$, *of V such that $J_r = \mathfrak{T}^{-1}L_r\mathfrak{T}$ $(0 \leqslant r \leqslant 3)$; $\mathfrak{T}$ is unique apart from a numerical multiplier.*

Proof. By Theorem 5 there is a basis $\mathfrak{B}$ in which $J_r \sim B_{2,r}$ $(0 \leqslant r \leqslant 3)$ and $\mathfrak{B}$ is unique apart from a numerical multiplier. If $L_r \sim M_r$ (in $\mathfrak{B}$) then (by Corollary 1) there is a matrix T, unique apart from a numerical multiplier, with $T^{-1}M_rT = B_r$. Hence the L.T. $\mathfrak{T}$ represented by T in $\mathfrak{B}$ is unique (apart from a numerical multiplier) in satisfying $\mathfrak{T}^{-1}L_r\mathfrak{T} = J_r$ $(0 \leqslant r \leqslant 3)$.

COROLLARY 3. *A regular* L.T. *A which anticommutes with all the L_r of Theorem* 5 *must be a numerical multiple of the involution $L_0L_1L_2L_3$.*

Proof. A has to satisfy $AL_rA^{-1} = -L_r$ $(0 \leqslant r \leqslant 3)$; the involution $L_0L_1L_2L_3$ certainly does this, and the result now follows from Corollary 2 with $J_r = -L_r$.

5.1. To illustrate the corollaries to Theorem 5, we prove that (cf. 3, p. 121) if N_0, N_1, N_2, N_3 are anticommuting 4×4 involution matrices, then there is a skew-symmetric matrix A such that $N_r = A^{-1}N_r'A$ $(0 \leqslant r \leqslant 3)$; A is readily computed.

Since $B_{2,r}'$ $(0 \leqslant r \leqslant 3)$ are anticommuting involutions, there is, by Corollary 1, a matrix T with

$$B_{2,r}' = TB_{2,r}T^{-1} \qquad (0 \leqslant r \leqslant 3),$$

and the columns of T, $\{\beta_1, \beta_2, \beta_3, \beta_4\}$, are found from $B_{2,0}\beta_1 = -iB_{2,3}B_{2,3}\beta_1 = \beta_1, \beta_2 = -iB_{2,3}B_{2,1}\beta_1, \beta_3 = B_{2,1}\beta_1$, $\beta_4 = -iB_{2,3}\beta_1$; these give $\beta_1 = e_3$, $\beta_2 = -e_1$, $\beta_3 = -e_4$, $\beta_4 = e_3$, that is

$$\text{(a)} \qquad T = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Since the N_r are anticommuting involutions, one could compute by Corollary 1 a matrix Q with

$$QN_rQ^{-1} = B_{2,r} \qquad (0 \leqslant r \leqslant 3).$$

We now have

$$QN_rQ^{-1} = (TB_{2,r}T^{-1})' = (TQN_rQ^{-1}T^{-1})', \text{ that is, } N_r = A^{-1}N_r'A,$$

where A, equal to $Q'T'Q$, is skew-symmetric because T is.

As a second illustration, we find a formula for all sets of four anticommuting 4×4 involution matrices with are skew-symmetric. This means (by Theorem 5) finding a formula for all matrices P which satisfy

$$PB_{2,r}P^{-1} = -(P^{-1})'B_{2,r}'P', \qquad (0 \leqslant r \leqslant 3);$$

using the illustration above, this means simply that $T^{-1}P'P$ anticommutes with $B_{2,r}$ $(0 \leqslant r \leqslant 3)$. By Corollary 3 this is equivalent to the statement that $T^{-1}P'P$ is a numerical multiple of

$$\begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}$$

which, by (a), means that $P'P$ is a numerical multiple of

$$\left(\begin{array}{cc:cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ \hdashline 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right)$$

that is, of $B_{2,1}$. Bearing in mind that the eigenvalues of $B_{2,1}$ are ± 1 it is easy to see that

$$B_{2,1} = \tfrac{1}{2}M'M, \text{ where } M = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ i & 0 & -i & 0 \\ 0 & i & 0 & i \end{pmatrix},$$

and hence that P has the form $c\Omega M$ where c is an arbitrary number and Ω an arbitrary orthogonal matrix (that is, $\Omega\Omega' = 1_4$). Thus the formula

$$J_r = \Omega M B_{2,r} M^{-1}\Omega' \qquad (0 \leqslant r \leqslant 3)$$

gives the required sets of involutions.

5.2. The result corresponding to (8) is as follows:

THEOREM 6. *Suppose L_0, L_1, L_2, L_3 are anticommuting involutions of V and $d(V) > 4$; then V is the direct sum of four-dimensional subspaces each stable for all the L_r, and in a suitable basis of V*

$$L_r \sim \text{diag.}(B_{2,r}, B_{2,r}, \ldots, B_{2,r}) \qquad (0 \leqslant r \leqslant 3).$$

Proof. Write S_1 for iL_2L_3 and S_2 for iL_3L_1. Then, as in the proof of Theorem 5, $\mathfrak{E}_1(L_0)$ is the direct sum of two-dimensional spaces, say $W_1, W_2, \ldots, W_q$, each of which is stable for S_1 and S_2 as well as for L_0. If W_r' is defined as $L_1(W_r)$, then by § 3(iv), $W_r' \subset \mathfrak{E}_{-1}(L_0)$ and W_r' is stable for S_1 and S_2. Thus the direct sum of W_r and W_r' is stable for L_0, L_1, S_1, and S_2; it is therefore stable for L_2 and L_3. It now follows from Theorem 5 that in this subspace of V there is a basis in which $L_s \sim B_{2,s}$. Since this holds for $1 \leqslant s \leqslant q$, and V is the direct sum of $\mathfrak{E}_1(L_0)$ and $\mathfrak{E}_{-1}(L_0)$, this completes the proof.

6. If a scalar product is defined in the spaces considered in Theorems 3 and 5, then the conclusions can be further particularized if the σ_r and the L_r are unitary (an involution is unitary if and only if it is hermitean). The modification is that the basis can be chosen orthonormal. This will follow in Theorem 3 from the fact that $\mathfrak{E}_1(\sigma_1)$ and $\mathfrak{E}_{-1}(\sigma_1)$ are orthogonal and that if $||\beta_1|| = 1$ then $||\beta_2|| = ||\sigma_2(\beta_1)|| = 1$. In the case of Theorem 5, the S_r defined in Theorem 6 will be unitary if the L_r are unitary, and consequently, as above β_1, β_2 can be chosen orthonormal in $\mathfrak{E}_1(L_0)$; it then follows from (b′) in Theorem 5 that β_3 and β_4 are orthonormal in $\mathfrak{E}_{-1}(L_0)$ while the two spaces $\mathfrak{E}_1(L_0)$ and $\mathfrak{E}_{-1}(L_0)$ are orthogonal. Similarly, if the matrices M_r in Theorem 1 are hermitean, then T can be chosen unitary. Since the $B_{2,r}$ are hermitean, it follows that the M_r are hermitean if and only if there is a unitary matrix U with $M_r = U^{-1}B_{2,r}U$ $(0 \leqslant r \leqslant 3)$.

7. In this section we generalize Theorem 5. Having defined

$$B_{1,0} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; B_{1,1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; B_{1,2} = iB_{1,0}B_{1,1} = i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

we now define, inductively, for every positive integer n a set of $2n+1$ matrices as follows:

$$B_{n,0} = \begin{pmatrix} 1_{2^{n-1}} & 0 \\ 0 & -1_{2^{n-1}} \end{pmatrix}, \quad B_{n,r} = \begin{pmatrix} 0 & B_{n-1,r-1} \\ B_{n-1,r-1} & 0 \end{pmatrix} (1 \leqslant r \leqslant 2n-1),$$

$$B_{n,2n} = i^n B_{n,0} B_{n,1} \ldots B_{n,2n-1}.$$

By § 3(i) it follows at once that, because the $B_{1,r}$ are involutions, the $2n+1$ involutions $B_{n,r}$ anticommute.

LEMMA.

$$(9) \qquad B_{n,2n} = i \begin{pmatrix} 0 & 1_{2^{n-1}} \\ -1_{2^{n-1}} & 0 \end{pmatrix}, \text{ that is, } B_{n,0} B_{n,1} \ldots B_{n,2n-1} B_{n,2n} = (-i)^n 1_{2^n}.$$

Proof. The equivalence of the two statements in (9) follows from the definition of $B_{n,2n}$ which gives $i^n B_{n,0} B_{n,1} \ldots B_{n,2n} = (B_{n,2n})^2 1_{2^n}$. The lemma is obvious when $n = 1$. Suppose $q > 1$ and that the lemma has been proved for $n = q-1$. By the definition of $B_{q,2q}$,

$$B_{q,2q} = i^q \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & B_{q-1,0} \\ B_{q-1,0} & 0 \end{pmatrix} \begin{pmatrix} 0 & B_{q-1,1} \\ B_{q-1,1} & 0 \end{pmatrix} \ldots \begin{pmatrix} 0 & B_{q-1,2q-2} \\ B_{q-1,2q-2} & 0 \end{pmatrix}$$

$$= i^q \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & B_{q-1,0} B_{q-1,1} \ldots B_{q-1,2q-2} \\ B_{q-1,0} B_{q-1,1} \ldots B_{q-1,2q-2} & 0 \end{pmatrix},$$

and by the induction hypothesis this gives

$$B_{q,2q} = i^q \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & (-i)^{q-1} 1_{2^{q-1}} \\ (-i)^{q-1} 1_{2^{q-1}} & 0 \end{pmatrix} = i \begin{pmatrix} 0 & 1_{2^{q-1}} \\ -1_{2^{q-1}} & 0 \end{pmatrix}.$$

THEOREM 7. *Suppose $L_0, L_1, \ldots, L_{2q-1}$ is a set of $2q$ anticommuting involutions in V and $d(V) = 2^q$; then, defining $L_{2q} = i^q L_0 L_1 \ldots L_{2q-1}$, there is a basis $\mathfrak{B}$ in which $L_r \sim B_{q,r}$ $(0 \leqslant r \leqslant 2q)$, and $\mathfrak{B}$ is unique apart from a numerical multiplier. The only regular* L.T.*'s of V which anticommute with $L_0, L_1, \ldots, L_{2q-1}$ are the numerical multiples of L_{2q}, and, of these, $\pm L_{2q}$ are the only involutions.*

Proof (by induction on q). Theorems 3 and 4 justify Theorem 7 when $q = 1$. Suppose the theorem is true when $q = n > 1$ and let $L_0, L_1, \ldots, L_{2n+1}$ be a set of $2n+2$ anticommuting involutions in V with $d(V) = 2^{n+1}$.

We first assume that V *has* a basis $\mathfrak{B}$ in which $L_r \sim B_{n+1,r}$ $(0 \leqslant r \leqslant 2n+2)$, and show that $\mathfrak{B}$ is essentially unique (that is, that any other basis with the same property must be $c\mathfrak{B}$). Define

$$L^*_s = iL_s L_{2n+2} \qquad (1 \leqslant s \leqslant 2n+1);$$

by § 3(v) the L^*_s are anticommuting involutions for which $\mathfrak{E}_1(L_0)$ and $\mathfrak{E}_{-1}(L_0)$ are stable; denote by L^{**}_s the L.T. of $\mathfrak{E}_1(L_0)$ effected by L^*_s. The involutions L^{**}_s anticommute in $\mathfrak{E}_1(L_0)$ which has dimension 2^n, and so, by the induction hypothesis, $\mathfrak{E}_1(L_0)$ has a basis $\mathfrak{B}_1 = \{\beta_1, \beta_2, \ldots, \beta_{2^n}\}$ (unique apart from a

numerical multiplier) in which $L^{**}{}_s \sim B_{n,s-1}$ $(1 \leqslant s \leqslant 2n)$. But, in the postulated basis $\mathfrak{B}$, $L_0 \sim I_{2^n}$, and

$$L_s^* \sim i\begin{pmatrix} 0 & B_{n,s-1} \\ B_{n,s-1} & 0 \end{pmatrix} i\begin{pmatrix} 0 & 1_{2^n} \\ -1_{2^n} & 0 \end{pmatrix} = \begin{pmatrix} B_{n,s-1} & 0 \\ 0 & B_{n,s-1} \end{pmatrix} \qquad (1 \leqslant s \leqslant 2n)$$

Hence $\beta_1, \beta_2, \ldots, \beta_{2^n}$ are the first 2^n members of $\mathfrak{B}$. Since

$$L_{2n+2} \sim i\begin{pmatrix} 0 & 1_{2^n} \\ -1_{2^n} & 0 \end{pmatrix}$$

(in $\mathfrak{B}$), it follows now that if $\mathfrak{B} = \{\beta_1, \beta_2, \ldots, \beta_{2^{n+1}}\}$ then

$$\beta_{r+2^n} = iL_{2n+2}\beta_r \qquad (1 \leqslant r \leqslant 2^n),$$

and so $\mathfrak{B}$ is determined completely (apart from a numerical multiplier) by $L_0, L_1, \ldots, L_{2n+1}$.

We now define $\mathfrak{B}'$ as $\{\mathfrak{B}_1, iL_{2n+2}\mathfrak{B}_1\}$ and proceed to prove that $L_r \sim B_{n+1,r}$ (in $\mathfrak{B}'$) for $0 \leqslant r \leqslant 2n+2$. Since $\mathfrak{B}_1$ spans $\mathfrak{E}_1(L_0)$ and $L_{2n+2}(\mathfrak{B}_1)$ spans $\mathfrak{E}_{-1}(L_0)$ (§ 3(iv)), it follows that in $\mathfrak{B}'$

$$L_0 \sim \begin{pmatrix} 1_{2^n} & 0 \\ 0 & -1_{2^n} \end{pmatrix}, \quad L_{2n+2} \sim i\begin{pmatrix} 0 & 1_{2^n} \\ -1_{2^n} & 0 \end{pmatrix} \text{ and}$$

$$iL_sL_{2n+2} \sim \begin{pmatrix} B_{n,s-1} & 0 \\ 0 & X_s \end{pmatrix} \qquad (1 \leqslant s \leqslant 2n),$$

where the exact form of X_s need not concern us. These imply

$$L_s = (L_sL_{2n+2})L_{2n+2} = \begin{pmatrix} B_{n,s-1} & 0 \\ 0 & X_s \end{pmatrix}\begin{pmatrix} 0 & 1_{2^n} \\ -1_{2^n} & 0 \end{pmatrix} = \begin{pmatrix} 0 & B_{n,s-1} \\ -X_s & 0 \end{pmatrix};$$

and since L_s anticommutes with L_0 it now follows by § 3(i) that

$$L_s \sim \begin{pmatrix} 0 & B_{n,s-1} \\ B_{n,s-1} & 0 \end{pmatrix} = B_{n+1,s} \qquad \text{for } s = 1, 2, \ldots, 2n, 2n+2.$$

To verify that the formula holds also when $s = 2n+1$, we note that $L_{2n+2} = i^{n+1}L_0 \ldots L_{2n+1}$ (by definition) and $L_{2n+2} \sim i^{n+1}B_{n+1,0}B_{n+1,1} \ldots B_{n+1,2n+1}$ from the known matrix representing L_{2n+2}. This proves $L_{2n+1} \sim B_{n+1,2n+1}$.

Finally, consider matrices M which satisfy

$$M^{-1}(-B_{q,s})M = B_{q,s} \qquad (0 \leqslant s \leqslant 2q-1). \tag{10}$$

The columns of such an M are, in order, the members of a basis (of the space of number 2^q-ples) in which the $(-B_{q,s})$ are represented by the $B_{q,s}$ respectively. Since the $2q$ involutions $-B_{q,1}, -B_{q,2}, \ldots, -B_{q,2q-1}$ anticommute, it follows from the first part of this theorem that such a basis is essentially unique. Since $M = B_{q,2q}$ satisfies (10), it now follows that $M = cB_{q,2q}$, with c an arbitrary number, is the complete solution of (10) and that M is an involution if and only if $c^2 = 1$. Since the L_r are represented by the $B_{q,r}$, this completes the proof of the theorem.

References

1. R. H. Good, *Properties of the Dirac matrices*, Rev. Mod. Phys., *27* (1955), 187.
2. P. A. M. Dirac, *The quantum theory of the electron*, Proc. Roy. Soc. A., *117* (1928), 616.
3. W. Pauli, *Contributions mathématiques à la théorie de Dirac*, Ann. Inst. Henri Poincaré, *6* (1936), 109.
4. B. L. Van der Waerden, *Die Gruppentheoretische Methode in der Quantenmechanik* (Berlin, 1932), 55.
5. A. S. Eddington, *On sets of anticommuting matrices*, J. London Math. Soc., *17* (1932), 58.
6. M. H. A. Newman, *Note on an algebraic theorem of Eddington*, J. London Math. Soc., *17* (1932), 93.
7. A. Hurwitz, *Ueber die Komposition der Quadratischen Formen*, Math. Annalen, *88* (1923), 1.

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FOURTH ORDER BOUNDARY VALUE PROBLEMS AND COMPARISON THEOREMS

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Introduction. This paper is primarily concerned with the existence of solutions of the fourth-order self-adjoint differential equation

$$[(r(x)y'')' + q(x)y']' - p(x)y = 0, \tag{1}$$

(where $r(x) > 0$, $q(x) \geqslant 0$, $p(x) \geqslant 0$ and all three coefficients are continuous on $[a, \infty)$) and one of the two-point boundary conditions:

$$y(a) = y'(a) = y(b) = y'(b) = 0, \tag{2}$$

or

$$y(a) = y'(a) = y_1(b) = y_2(b) = 0; \tag{3}$$

the subscript notation for any solution $y(x)$ denoting:

$$y_1(x) = r(x)y''(x) \quad \text{and} \quad y_2(x) = y_1'(x) + q(x)y'(x). \tag{4}$$

If $q(x)$ is differentiable then (1) is equivalent to the more familiar form: $(r(x)y'')'' + (q(x)y')' - p(x)y = 0$ and, in either case, solutions exist with the appropriate derivatives. In particular, attention is given to establishing criteria for the existence of a smallest number $b \in (a, \infty)$ for which (2), or (3), is satisfied by a non-trivial solution of (1).

DEFINITION 1. *The number $\eta_1(a)$ is the smallest number $b \in (a, \infty)$ such that* (2) *is satisfied non-trivially by a solution of* (1).

DEFINITION 2. *The number $\mu_1(a)$ is defined in the same way by* (3).

For $q \equiv 0$, $\eta_1(a)$ is the first conjugate point of $x = a^*$ as established by Leighton and Nehari **(6)** and for general q, $\eta_1(a)$ is recognized here to be the first conjugate point of a second-order vector-matrix equation in the usual sense. Also $\mu_1(a)$ reduces to the analogy to focal points for second-order scalar equations introduced by the author **(3)**. Other recent investigations for the case of $q \equiv 0$ and $p > 0$ have been made by Howard **(5)**, making extensive use of Rayleigh quotients, and by the author **(4)**, utilizing the simple vector-matrix formulation

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*The minimum $b > a$ such that a non-trivial solution of (1) has four zeros on $[a, b]$.

$$\begin{pmatrix} y \\ y_1 \end{pmatrix}'' = \begin{pmatrix} 0 & 1/r \\ p & 0 \end{pmatrix}\begin{pmatrix} y \\ y_1 \end{pmatrix}$$

and comparison with the special scalar equation

$$(y''/p(x))'' - p(x)y = 0,\ p(x) > 0.$$

It is also known **(6,** Theorems 12.1 and 12.2**)** that under certain conditions the middle term of (1) can be removed and the known theory of $q \equiv 0$ can be applied to the resulting equation. However, there are many cases where such a transformation cannot be made into an equation where $p(x)$ does not change sign, in order to apply the results of **(6),** or where $p(x)$ is positive, as required by **(3; 4; 5).** Also, in most of the above-mentioned studies, there is a strong dependence on the monotonicity of certain solutions and their derivatives. It is easy to see that for $q > 0$ such properties are not always available.

A systems formulation of (1), which is not as obvious but more useful than the above-mentioned one of the author **(4),** was introduced by Sternberg and Sternberg **(10),** prior to all of the papers previously mentioned here. They investigated fourth-order matrix equations of the form of (1), including the middle term and found what amounts to sufficient conditions on the coefficients to insure the existence of $\eta_1(a)$ (see concluding comment of § 2).

It is this paper which provides the systems formulation of (1) to be used throughout the present discussion. Following earlier fundamental results of Sternberg **(10),** they utilized one of his canonical systems, involving a variational multiplier and functional side conditions. Here, use will be made of his other (equivalent) canonical system and the need for monotonicity of solutions of (1) is avoided. An elementary self-contained derivation of this latter system will now be given.

Using any solution $y(x)$ of (1) and the notation (4), the vectors

$$\alpha(x) = \begin{pmatrix} y \\ y' \end{pmatrix} \quad \text{and} \quad \tilde{\alpha}(x) = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

satisfy the system*

$$(5)\quad \begin{cases} \alpha' = A\alpha + B\tilde{\alpha} \\ \tilde{\alpha}' = C\alpha + A\tilde{\alpha} \end{cases} \text{ where } A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ B = \frac{1}{r}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ \text{and } C = \begin{pmatrix} 0 & -q \\ p & 0 \end{pmatrix}.$$

This system can be simplified by the substitution

$$\alpha = D(x)\beta,\ \tilde{\alpha} = \tilde{D}(x)\tilde{\beta}$$

where $D(x)$ and $\tilde{D}(x)$ are both solutions of $D' = AD$ and the respective initial conditions:

$$D(a) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I, \qquad \tilde{D}(a) = C_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

*For a similar formulation, see Coddington and Levinson, *Theory of Ordinary Differential Equations*, p. 207, problem 19, replacing $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ by $\begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}$.

Then

$$D(x) = \begin{pmatrix} 1 & x-a \\ 0 & 1 \end{pmatrix}, \qquad \bar{D}(x) = D(x)C_0$$

and (5) reduces to the second canonical system of Sternberg **(8, (2.4))**:

$$(6) \qquad \begin{cases} \beta' = E(x)\tilde{\beta} \\ \tilde{\beta}' = -F(x)\beta \end{cases} \quad \text{where} \quad \begin{aligned} E(x) &= D^{-1}B\bar{D} = \frac{1}{r(x)}E_1(x-a) \\ F(x) &= -\bar{D}^{-1}CD = p(x)F_1(x-a) + q(x)Q_1. \end{aligned}$$

and

$$E_1(x) = \begin{pmatrix} x^2 & -x \\ -x & 1 \end{pmatrix}, \quad F_1(x) = \begin{pmatrix} 1 & x \\ x & x^2 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The initial value $\bar{D}(a) = C_0$ was chosen so that the resulting matrix coefficients E and F of (6) would be symmetric. Fortunately, they are also positive semi-definite, that is, $E(x) \geqslant 0$, $F(x) \geqslant 0$ and if $p(x)\,q(x) > 0$ then $F(x) > 0$, that is, positive definite. Since the determinant of $E(x) = \det E(x) \equiv 0$ then (6) cannot be expressed as a simple second-order vector-matrix equation in β (that is, $(R(x)\beta')' + P(x)\beta = 0$, $R > 0$, $P \geqslant 0$) but it is found that much of the known theory of such equations is also applicable to first-order systems of the type (6), particularly the use of matrix Riccati equations **(1; 7)**.

The techniques and results of this paper are direct analogies of those applicable to the scalar system: $y' = e(x)z$, $z' = -f(x)y$, where $e(x) \geqslant 0$ and $f(x) \geqslant 0$, and the second-order scalar equation $(r(x)y')' + p(x)y = 0$, with non-negative coefficients **(2)**.

First of all, conditions for the existence of $\mu_1(a)$ will be established; secondly, $\mu_1(a)$ will be assumed to exist and further requirements on the coefficients of (1) will be added to insure the existence of $\eta_1(a)$; thirdly, these will be combined to give direct criteria for $\eta_1(a)$ and, finally, comparison theorems between two equations of type (1) will be established.

Throughout the paper, when proofs involve $E(x)$ and $F(x)$, $x = a$ will be taken to be $x = 0$ for simplicity, but no loss of generality. Also, subscript notation (4) and the inequality notation denoting definiteness of matrices will be adopted as standard notation.

1. Relations between system (4) and equation (1). Consider two fundamental solutions $u(x)$ and $v(x)$ of (1) satisfying, respectively, the initial conditions:

$$(7) \qquad u(a) = u'(a) = 0,\ u_1(a) = 1,\ u_2(a) = 0$$

and

$$(8) \qquad v(a) = v'(a) = v_1(a) = 0,\ v_2(a) = 1.$$

Any solution of (1) which has a double zero at $x = a$ must be a linear combination of $u(x)$ and $v(x)$ and the numbers $\mu_1(a)$ and $\eta_1(a)$ are the smallest zeros on (a, ∞) of, respectively, the sub-wronskians:

$$\rho(x) = \rho[u; v] = u_1 v_2 - v_1 u_2 \tag{9}$$

and

$$\sigma(x) = \sigma[u; v] = uv' - vu'. \tag{10}$$

Recalling the transformation of the opening section of equation (1) into the system (6) let

$$\mathfrak{Y}(x) = \begin{pmatrix} u & v \\ u' & v' \end{pmatrix}, \qquad \tilde{\mathfrak{Y}} = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} \tag{11}$$

$$Y(x) = D^{-1}(x)\mathfrak{Y}(x), \ \tilde{Y}(x) = \tilde{D}^{-1}(x)\tilde{\mathfrak{Y}}(x). \tag{12}$$

Then Y and $\tilde{Y}$ satisfy the matrix system

$$\begin{aligned} Y' &= E(x)\tilde{Y}, \ Y(a) = 0 \\ \tilde{Y}' &= -F(x)Y, \ \tilde{Y}(a) = C_0. \end{aligned} \tag{13}$$

And, furthermore,

$$\begin{aligned} \det Y(x) &= \det \mathfrak{Y}(x) = \sigma(x), \qquad \sigma(a) = 0, \\ \det \tilde{Y}(x) &= \det \tilde{\mathfrak{Y}}(x) = \rho(x), \qquad \rho(a) = 1. \end{aligned} \tag{14}$$

Therefore, the existence of $\eta_1(a)$ and $\mu_1(a)$ become problems of finding (matrix) singularities of components of solution pairs of (13).

2. Conditions for existence of $\mu_1(a)$. For $q \equiv 0$ conditions insuring the existence of $\mu_1(a)$ have already been given, the strongest of which is:

THEOREM 2.1 **(3).** *If $q = 0$ and*

$$\int^{\infty} p(x)(\overset{z}{I}{}^2 p)^2 dx = \infty$$

then $\mu_1(b)$ exists for every $b \in [a, \infty)$, where $\overset{z}{I}{}^2 p$ is the iterated antiderivative

$$\int^{z} \int^{t} p(s)ds\, dt.$$

For $p \equiv 0$ there is a striking relation between focal conditions for second-order equations and the existence of $\mu_1(a)$. Let $\bar{\mu}_1(a)$ be the smallest $b \in (a, \infty)$ for which a non-trivial solution of

$$(r(x)y')' + q(x)y = 0 \ (r \text{ and } q \text{ from } (1)) \tag{15}$$

satisfies the focal conditions

$$y(a) = y'(b) = 0. \tag{16}$$

Then it is a simple matter to check the following:

THEOREM 2.2. *If $p(x) \equiv 0$ then $\mu_1(a)$ exists if and only if $\bar{\mu}_1(a)$ exists and, furthermore, $\mu_1(a) = \bar{\mu}_1(a)$.*

Comparison theorems of § 4 will extend these theorems to the cases where $q(x) \geqslant 0$ and $p(x) \geqslant 0$, respectively, but first, consider the complete equation (1) and the matrix system (13).

Since $\rho(a) = 1$ there is a largest number $b, a < b \leqslant \infty$, such that $\tilde{Y}(x)$ is non-singular on $[a, b)$. On this interval let $K(x) = Y(x)\tilde{Y}^{-1}(x)$ then $K(x)$ satisfies the matrix Riccati equation and initial condition:

$$K' = E(x) + KF(x)K, \qquad K(a) = 0, \tag{17}$$

and $K(x)$ is symmetric on $[a, b)$. (For these and other properties of $K(x)$ see **(1; 7)**.)

Let $\xi = (\xi_i)$ be a non-zero constant (column) vector; then by

$$\xi^* K(x)\xi \geqslant \int_a^x \xi^* E(t)\xi\, dt = \int_a^x (t\xi_1 - \xi_2)^2 \frac{1}{r(t)}\, dt \qquad (a = 0), \tag{17}$$

$K(x)$ is positive-definite on (a, b), and $\det K(x) = \sigma(x)/\rho(x) > 0$, giving $\sigma(x) > 0$. Hence, $Y(x)$ is non-singular, $\eta_1(a)$ does not exist on (a, b) and since $\mu_1(a) \neq \eta_1(a)$ (see Lemma 2.1), then:

THEOREM 2.3. If $\eta_1(a)$ exists then $\mu_1(a)$ exists and $a < \mu_1(a) < \eta_1(a)$.*

The definiteness of $K(x)$ established above yield further information which will be used in succeeding sections. Recall from (12) that

$$K = Y\tilde{Y}^{-1} = D^{-1}\mathfrak{Y}\tilde{\mathfrak{Y}}^{-1}DC_0, \qquad C_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

A simple computation verifies that $DC_0 = C_0D^{*-1}$ and, hence,

$$DKD^* = \mathfrak{Y}\tilde{\mathfrak{Y}}^{-1}C_0 \tag{18}$$

where (again using $a = 0$)

$$DKD^* = \begin{pmatrix} k_{11} + 2xk_{12} + x^2k_{22} & k_{12} + xk_{22} \\ k_{12} + xk_{22} & k_{22} \end{pmatrix}, \quad \mathfrak{Y}\tilde{\mathfrak{Y}}^{-1}C_0 = \frac{1}{\rho}\begin{pmatrix} r\sigma' & r \\ r & r' \end{pmatrix}$$

and, as in **(3)**, $r(x) = u'v_1 - v'u_1 (\equiv uv_2 - vu_2)$. Since $K(x) > 0$ on (a, b) then $\sigma' > 0$ and $r' = \rho(x)k_{22}(x) > 0$. This together with the initial-value $r(a) = 0$ yields

LEMMA 2.1. *If $\rho(x) > 0$ on $[a, b), a < b \leqslant \infty$, then $\sigma(x) > 0$, $\sigma'(x) > 0$, $r'(x) > 0$ and $r(x) > 0$ on $(a, b]$ and, furthermore,*

$$r(x)/\rho(x) = K_{12}(x) + (x - a)K_{22}(x) > 0 \text{ on } (a, b).$$

Finally, the symmetry and definiteness of $K(x)$ yields that $\operatorname{tr} K(x) > 0$, $(\operatorname{tr} K(x))^2 > 4 \det K(x) = \sigma(x)/\rho(x)$ on (a, μ_1) and, hence, that:

LEMMA 2.2. *If $\mu_1(a)$ exists then $\operatorname{tr} K(x) \to \infty$ as $x \to \mu_1(a)$ on $(a, \mu_1(a))$.*

*Also true regardless of the signs or changes of sign of $q(x)$ and $p(x)$, as can be seen by a more careful examination of Lemma 2.1 and its consequences.

Now, suppose that $\mu_1(a)$ does not exist, that is, $b = \infty$ in the above discussion. Then $K(x)$ is defined and non-singular on (a, ∞) and by (17)

$$K^{-1}K'K^{-1} \geq F \text{ (the difference being positive semi-definite).}$$

If $x_0 \in (a, \infty)$ and $\xi = (\xi_i)$ is a constant non-zero vector then on $[x_0, \infty)$

$$\xi^* K^{-1}(x_0)\xi > \xi^* K^{-1}(x_0)\xi - \xi^* K^{-1}(x)\xi > \int_{x_0}^{x} (\xi_1 + t\xi_2)^2 p(t)dt + \int_{x_0}^{x} \xi_2^2 q(t)dt$$

and the following is an immediate consequence:

THEOREM 2.4. *If either*

$$\int^{\infty} x^2 p(x)dx = \infty \quad \text{or} \quad \int^{\infty} q(x)dx = \infty$$

then $\mu_1(b)$ exists for every $b \in [a, \infty)$.

Note that for $q(x) \equiv 0$ this is a corollary of Theorem 2.1. Also, for $q \equiv 0$ it is known **(3; 5)** that the existence of $\mu_1(a)$ combined with $\int^{\infty} 1/r = \infty$ give the existence of $\eta_1(a)$. This will be established in the next section for for $q \geq 0$. Sternberg and Sternberg **(10)** proved this result for the special case of $r(x)$ bounded, but for matrix equations.

3. Conditions for the existence of $\eta_1(a)$. Assume that $\mu_1(a)$ exists but that $\eta_1(a)$ does not, that is, $\det Y(x) = \sigma(x) > 0$ on (a, ∞). Let $H = -\bar{Y}Y^{-1}$ then $H(x)$ is defined on (a, ∞), is symmetric, and

$$H' = F(x) + HE(x)H. \tag{19}$$

Also on (a, μ_1), $H(x) = -K^{-1}(x)$ and is negative definite. Since $H(\mu_1)$ is singular then by (19) the maximum eigenvalue of $H(x)$ (written max. e.v. $H(x)$) is a non-decreasing function on (a, ∞) and max. e.v. $H(\mu_1) = 0$. Suppose that max. e.v. $H(x) \equiv 0$ on (μ_1, ∞), then $\det H(x) \equiv 0$ and $\rho(x) \equiv 0$ on that interval. This is possible if $p \equiv q \equiv 0$ on (μ_1, ∞) and it will now be shown that this is also necessary. Since $H'(x) \geq F(x) \equiv 0$ and there is a non-zero vector $\xi = (\xi_i)$ such that $H(\mu_1)\xi = 0$ then on (μ_1, ∞)

$$0 \geq \xi^* H(x)\xi \geq \int_{\mu_1}^{x} (\xi_1 + t\xi_2)^2 p(t)dt + \int_{\mu_1}^{x} (\xi_2^2 q(t)dt \geq 0$$

implying $p(x) \equiv 0$ and either $q(x) \equiv 0$ on (μ_1, ∞) or $\xi_2 = 0$. If $\xi_2 = 0$ then

$$\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \xi_1 \neq 0$$

and $H(\mu_1)\xi = 0$ gives $h_{11}(\mu_1) = 0$ and $h_{12}(\mu_1) = 0$. By (19) $h_{11}(x)$ is non-decreasing, $h_{11}(x) \leq$ max. e.v. $H(x) = 0$, and

$$h_{11}' = \frac{1}{r(x)} (xh_{11} - h_{12})^2 \text{ on } [a, \infty) \qquad (a = 0).$$

Using the fact that $H = -K^{-1}$ on (a, μ_1) then by Lemma 2.1

$$xh_{11} - h_{12} = \frac{xk_{22} + k_{12}}{\det K} = \frac{\tau/\rho}{\sigma/\rho} \to \frac{\tau(\mu_1)}{\sigma(\mu_1)} > 0 \text{ as } x \to \mu_1.$$

Therefore $h_{11}'(\mu_1) > 0$, which contradicts that $h_{11} \equiv 0$ on (μ_1, ∞). Hence, $q(x) \equiv p(x) \equiv 0$ on $[\mu_1, \infty)$, if max. e.v. $H(x) \equiv 0$ on $[\mu_1, \infty)$. (Note that if $p(x) \equiv q(x) \equiv 0$ on $[\mu_1, \infty)$ then $\rho(x) \equiv 0$, $\sigma(x) \equiv \sigma(\mu_1) > 0$ and $\det H(x) \equiv 0$ on $[\mu_1, \infty)$.) Suppose that $p(x) + q(x) \not\equiv 0$ on (X, ∞) for every $X \in (a, \infty)$. Then there is a number $x_1 \in (\mu_1, \infty)$ such that max. e.v. $H(x) > 0$ on $[x_1, \infty)$.

Suppose that min. e.v. $H(x) < 0$ on $[x_1, \infty)$, as it is on (a, μ_1). Then $\det H(x) < 0$ and $H(x)$ is non-singular on $[x_1, \infty)$. From (19), $H^{-1}H'H^{-1} \geqslant E(x)$ and if $\xi = (\xi_i)$ is any constant vector of length $|\xi| = 1$, then on $[x_1, \infty)$

$$\xi^* H^{-1}(x_1)\xi - \xi^* H^{-1}(x)\xi \geqslant \xi^* \left(\int_{x_1}^{x} E \right) \xi \tag{20}$$

and

$$\xi^* H^{-1}(x)\xi \leqslant \text{max. e.v. } H^{-1}(x_1) - \text{min. e.v. } \int_{x_1}^{x} E. \tag{21}$$

LEMMA 3.1. *If* $\int_1^\infty 1/r = \infty$ *then* min. e.v. $\int_{x_1}^{x} E \to \infty$ *as* $x \to \infty$.

Proof. Let $a = 0$ and compute

$$\det \int_{x_1}^{x} E = \int_{x_1}^{x} t^2/r \int_{x_1}^{x} 1/r - \left(\int_{x_1}^{x} t/r \right)^2 \geqslant 0 \quad \text{on} \quad [x_1, \infty).$$

Then $(\det \int_{x_1}^{x} E)' = L(x)/r(x)$, where $L(x) = \int_{x_1}^{x} t^2/r + x^2 \int_{x_1}^{x} 1/r - 2x \int_{x_1}^{x} t/r > 0$ on (x_1, ∞). Furthermore, $L'(x) = 2(x \int_{x_1}^{x} 1/r - \int_{x_1}^{x} t/r) > 0$ and $L''(x) = 2 \int_{x_1}^{x} 1/r > 0$ and $\to \infty$ as $x \to \infty$. Successive applications of L'Hopital's Rule yield

$$\text{min. e.v.} \int_{x_1}^{x} E \geqslant \left(\det \int_{x_1}^{x} E \Big/ \operatorname{tr} \int_{x_1}^{x} E \right) = \int_{x_1}^{x} \frac{L(t)}{r(t)} dt \Big/ \int_{x_1}^{x} \frac{t^2 + 1}{r(t)} dt \to \infty$$

as $x \to \infty$

and the lemma is proved.

Add to the main argument that $\int^\infty 1/r = \infty$ and by Lemma 3.1 and (21) there is a number $x_2 \in (x_1, \infty)$ such that on $[x_2, \infty)$ both $H^{-1}(x)$ and $H(x)$ are negative definite. But this contradicts that max. e.v. $H(x) > 0$ and it follows that there exists a smallest number $\mu_2 \in [\mu_1, \infty)$ such that min. e.v. $H(\mu_2) = 0$. Suppose that min. e.v. $H(x) \equiv 0$ on $[\mu_2, \infty)$. Then $\det H \equiv 0$ and $H \geqslant 0$ on this interval. Let $\bar{x} \in (\mu_2, \infty)$ such that $p(x) + q(x) \not\equiv 0$ on $[\mu_2, \bar{x}]$. Then there is a (constant) vector $\bar{\xi}$ such that $H(\bar{x})\bar{\xi} = 0$. Therefore, by use of (19):

$$0 \geqslant \bar{\xi}^* H(\mu_2)\xi \geqslant \bar{\xi}^* \int_{\mu_2}^{\bar{x}} F\bar{\xi} > 0,$$

which implies that

$$\int_{\mu_2}^{\bar{x}} (\bar{\xi}_1 + t\bar{\xi}_2)^2 p(t)dt + \bar{\xi}_2^2 \int_{\mu_2}^{\bar{x}} q(t)dt = 0, \quad p(t) \equiv 0 \text{ on } [\mu_2, \bar{x}] \text{and } \bar{\xi}_2 = 0.$$

Therefore $h_{11}(\bar{x}) = 0$ and this, together with $h_{11}' \geqslant 0$ and $h_{11}(x) \leqslant 0$ on $[\mu_2, \bar{x}]$, gives $h_{11} \equiv 0$ and $h_{12}(x) \equiv 0$, since $\det H(x) \equiv 0$, on this interval. Equation (19) then becomes

$$h_{22}' = q(x) + \frac{1}{r(x)} h_{22}^2 \text{ on } [\mu_2, \bar{x}].$$

Since $q(x) \not\equiv 0$ on $[\mu_2, \bar{x}]$ then there is a number $\bar{x}_1 \in (\mu_2, \bar{x})$ such that $h_{22}(x) > 0$ on $[\bar{x}_1, \bar{x}]$. Then $h_{22}^{-2} h_{22}' \geqslant 1/r(x)$ and

$$\int_{\bar{x}_1}^{\bar{x}} 1/r < \frac{1}{h_{22}(\bar{x}_1)}.$$

This is true for any $\bar{x} > \bar{x}_1$, which contradicts $\int^{\infty} 1/r = \infty$ and there exists a number $x_3 \in (\mu_2, \infty)$ such that min. e.v. $H(x) > 0$, that is, $H(x) > 0$ and $H^{-1}(x) > 0$, on $[x_3, \infty)$. This being the case, inequality (20) holds for x_1 replaced by x_3 and

$$\xi^* H^{-1}(x_3)\xi > \int_{x_3}^{x} \xi^* E(t)\xi dt = \int_{x_3}^{x} (t\xi_1 - \xi_2)^2 \frac{1}{r(t)} dt \qquad (a = 0)$$

which contradicts the assumption that $\int^{\infty} 1/r = \infty$. Since all consequences of the assumption that $\eta_1(a)$ does not exist are eliminated then:

THEOREM 3.1. *If $\mu_1(a)$ exists, $\int^{\infty} 1/r = \infty$, and $p(x)$ and $q(x)$ are not both identically zero for large x then $\eta_1(a)$ exists.* (See Theorem 4.2.)

THEOREM 3.2. *If $\eta_1(a)$ exists then* max. e.v. $H(x) \to \infty$, *as $x \to \eta_1(a)$ on $(a, \eta_1(a))$.*

Proof. Suppose that max. e.v. $H(x) \to \bar{h} < \infty$ as $x \to \eta_1(a)$. Then for any constant vector ξ, the non-decreasing scalar $\xi^* H(x)\xi$ has a limit as $x \to \eta_1(a)$. By taking, in turn,

$$\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

it follows readily that each component of $H(x)$ has a limit as $x \to \eta_1(a)$. Note that on (a, η_1): $Y' = E\bar{Y} = (-EH)Y$, $\sigma(x) = -(\operatorname{tr} EH)\sigma(x)$ and $\sigma(x) = \sigma(x_1) \exp\{-\int_{x_1}^{x_2} \operatorname{tr} EH\}$, where $x \in (a, \eta_1)$. But $H(x)$ hasa limit as $x \to \eta_1(a)$ which implies that $\sigma(\eta_1) > 0$, a contradiction, thus proving the theorem.

Finally, if the zeros of $\rho(x)$ are isolated, let $\mu_i(a)$ be its ith zero on (a, ∞) and, in general, let $\mu_{i+1}(a)$ be the smallest zero of $\rho(x)$ on (μ_i, ∞) such that $\rho(x) \not\equiv 0$ on (μ_i, μ_{i+1}).

Proof of the following theorems are omitted since they follow readily from the preceding analysis.

THEOREM 3.3. *If* $\mu_1(a)$ *and* $\mu_2(a)$ *both exist and* $\int^\infty x^2/r(x)dx = \infty$, *then* $\eta_1(a)$ *exists.*

THEOREM 3.4. *If* $\mu_1(a)$, $\mu_2(a)$, *and* $\mu_3(a)$ *all exist, then* $\eta_1(a)$ *exists.*

THEOREM 3.5. *If* $\eta_1(a)$ *does not exist and* $\int^\infty p = \infty$ *then* $\mu_1(a)$ *and* $\mu_2(a)$ *exist.*

THEOREM 3.6. *If*

$$\int^\infty \frac{x^2}{r(x)}\,dx = \infty$$

and $\int^\infty p = \infty$ *then* $\eta_1(c)$ *exists for all* $c \in [a, \infty)$.

By letting $p \equiv 0$ and $b = \eta_1(a)$, Theorems 2.4 and 3.1 yield, as an immediate corollary; the second-order result:

THEOREM 3.7. *If* $\int^\infty 1/r = \infty$ *and* $\int^\infty p = \infty$ *then either* (15) $(rz')' + qz = 0$ *or* $(rz')' + qz = 1$ *has a non-trivial solution* $z(x)$ *and a number* $b > a$ *such that*

$$z(a) = z(b) = 0 \quad \text{and} \quad \int_a^b z(x)dx = 0.$$

4. Comparison theorems for $\mu_1(a)$. Consider a second equation of type (1):

$$(1_0) \qquad [(r_0(x)y'')' + q_0(x)y']' - p_0(x)y = 0$$

and denote the various quantities defined for (1) in the preceding sections by the same symbols with an additional subscript or superscript "0," that is,

$$u_0, v_0, \sigma_0, \phi_0, Y_0, \bar{Y}_0, \mu_1^0(a), \eta_1^0(a), K_0, \text{ etc.}$$

THEOREM 4.1. *If* $0 < r(x) \leqslant r_0(x)$, $0 \leqslant p_0(x) \leqslant p(x)$, $0 \leqslant q_0(x) \leqslant q(x)$ *and* $\mu_1{}^0(a)$ *exists, then* $\mu_1(a)$ *exists and* $\mu_1(a) \leqslant \mu_1{}^0(a)$. *Furthermore,* $\mu_1(a) = \mu_1{}^0(a)$ *if and only if*

$$(22) \qquad r_0(x) \equiv r(x),\ q_0(x) \equiv q(x),\ p_0(x) \equiv p(x)$$

on $(a, \mu_1{}^0(a)]$.

Proof. Suppose that $\mu_1{}^0(a)$ exists but that $\mu_1(a)$ does not on $[a, \mu_1{}^0]$. Then on $(a, \mu_1{}^0)$: $K(x)$ and $K_0(x) = Y_0\bar{Y}_0^{-1}$ are defined, symmetric, and

$$K' = E + KFK, \qquad K_0' = E_0 + K_0F_0K_0.$$

Subtraction of these equations and adding and subtracting terms yields

$$(23) \quad (K - K_0)' - (K_0F_0)(K - K_0) - (K - K_0)(F_0K_0) = E - E_0 + K(F - F_0)K + (K - K_0)F_0(K - K_0),$$

where

$$(24) \qquad E(x) - E_0(x) = \left[\frac{1}{r(x)} - \frac{1}{r_0(x)}\right] E_1(x-a),$$

$$F(x) - F_0(x) = [p(x) - p_0(x)]F_1(x-a) + [q(x) - q_0(x)]Q_1.$$

Therefore, each of the three terms on the right-hand side of (23) is positive semi-definite and the fact that $K - K_0 \geqslant 0$ may be obtained by "solving" (23) as a linear non-homogeneous matrix differential equation, as in **(1)** using the techniques of **(7)**. On $[a, \mu_1^0)$ let $K - K_0 = J(x)L(x)J^*(x)$, where $J' = (K_0F_0)J$ and $J(a)$ is non-singular, and substitution into (23) gives $L(a) = 0$ and

$$(25) \qquad L'(x) = J^{-1}\{(E - E_0) + K(F - F_0)K + (R - R_0)F_0(K - K_0)\}J^{*-1}.$$

One appropriate solution for $J(x)$ is $J = \bar{Y}_0^{*-1}$ and, hence,

$$(25') \qquad L(x) = \int_a^x \bar{Y}_0^*\{(E - E_0) + K(F - F_0)K + (K - K_0)F_0(K - K_0)\}\bar{Y}_0,$$

which is positive semi-definite on (a, μ_1^0). It follows that $K \geqslant K_0$ and the fact that $\operatorname{tr} K(x) \geqslant \operatorname{tr} K_0(x)$, together with Lemma 2.1, contradicts the continuity of $K(x)$ on the closed interval $[a, \mu_1^0]$ and the first part of Theorem 4.1 is proved and $\mu_1(a) \leqslant \mu_1^0(a)$.

The proof of the remaining part is more difficult. Suppose that $\mu_1(a) = \mu_1^0(a)$, then $\bar{Y}(x)$ and $\bar{Y}_0(x)$ have their first singularity simultaneously, as do $H = -\bar{Y}Y^{-1}$ and $H_0 = -\bar{Y}_0Y_0^{-1}$. Note that by Theorem 2.3, $\eta_1(a)$ and $\eta_1^0(a)$ do not appear on $[a, \mu_1]$ and $H(x)$ and $H_0(x)$ are defined on $(a, \mu_1]$, and on (a, μ_1), $H = -K^{-1}$, $H_0 = -K_0^{-1}$. Hence $H(x)$ and $H_0(x)$ are both negative semi-definite on (a, μ_1).

Throughout the remainder of this paper there will be a repeated need to relate the definiteness of $H - H_0$ to that of $K - K_0$ The following general (any order) matrix theorem is useful.

LEMMA M. *If A and B are two non-singular matrices, $A - B$ is positive definite (semi-definite) and B is positive definite then $B^{-1} - A^{-1}$ is positive definite (semi-definite).*

The proof follows readily from the fact (see Bocher, *Introduction to Higher Algebra* [New York, 1907], p. 171) that A and B may be diagonalized simultaneously, and, in fact, all eigenvalues of B may be transformed to the value "1."

Returning to the main argument, $K - K_0 \geqslant 0$ implies $H - H_0 \geqslant 0$ and on $(a, \mu_1]$; as in the discussion for K and K_0.

$$H' = F + HEH, \qquad H_0' = F_0 + H_0E_0H_0,$$

and

$$(26) \qquad (H - H_0)' - (H_0E_0)(H - H_0) - (H - H_0)(E_0H_0) = F - F_0 + H(E - E_0)H + (H - H_0)E_0(H - H_0).$$

Let $H - H_0 = J_1(x)M(x)J^*_1(x)$, where $J_1' = (H_0E_0)J_1$. Then $J_1 = Y_0^{*-1}$ is a multiplying factor and

$$M'(x) = Y_0^*\{(F \quad F_0) + H(E - E_0)H + (H - H_0)E_0(H - H_0)\} \qquad (27)$$
$$Y_0 \geqslant 0 \text{ on } (a, \mu_1].$$

Since $H_0(\mu_1)$ is singular, there is a non-zero vector ξ_0 such that $H_0(\mu_1)\xi_0 = 0$ and by letting $x \to \mu_1$,

$$0 \leqslant \xi_0^*[H(\mu_1) - H_0(\mu_1)]\xi_0 = \xi_0^*H(\mu_1)\xi_0 \leqslant 0$$

which implies that min. e.v. $(H - H_0)(\mu_1) = 0$ and min. e.v. $M(\mu_1) = 0$. Therefore, there is a non-zero vector ξ such that $\xi^*M(\mu_1)\xi = 0$ and since $H - H_0 \geqslant 0$ then so is $M(x)$ and $\xi^*M(x)\xi \geqslant 0$ on $(a, \mu_1]$. But by (27) $(\xi^*M(x)\xi)' \geqslant 0$ which implies that $\xi^*M(x)\xi = \xi^*M'(x)\xi = 0$ on $(a, \mu_1]$ and min. e.v. $M(x)$ = Min. e.v. $(H - H_0)$ = min. e.v. $(K - K_0)$ = min. e.v. $L(x)$ = 0 on (a, μ_1).

Suppose there is a subinterval $I = [x_1, x_2]$ of (a, μ_1) such that $r(x) < r_0(x)$ on I. Let ξ be a non-zero vector such that $\xi^*M(x_2)\xi = 0$. Since the integrand of (25) is the sum of the three exhibited positive semi-definite terms then

$$\int_a^{x_2} \xi^* Y_0^*(E - E_0)Y_0\xi = 0 \text{ and } \eta^*(E - E_0)\eta = 0 \text{ on } (a, x_2),$$

where $\eta(x) = Y_0(x)\xi \not\equiv 0$.

Let $a = 0$, then $\eta^*E_1(x)\eta = (x\eta_1 - \eta_2)^2 = 0$ and $\eta_2 = x\eta_1$ on I. Recalling (13) and that $K_0 = Y_0Y_0^{-1}$, $\eta(x)$ satisfies

$$\eta' = -(F_0K_0)\eta \text{ on } (a, \mu_1) \text{ and } \eta(x) = \begin{pmatrix} 1 \\ x \end{pmatrix}\eta_1(x), \ \eta_1 \not\equiv 0 \qquad (a = 0).$$

That these are incompatible is seen by substituting the latter into the former and by multiplying on the left by the (row) vector $(-x, 1)$, which yields on I: (letting $K_0 = (k_{ij}^0)$)

$$1 = -(-x, 1)(F_0, K_0)\begin{pmatrix} 1 \\ x \end{pmatrix} = -q_0(k_{12}^0 + xk_{22}^0) \qquad (a = 0).$$

But $q_0 \geqslant 0$ and by Lemma 2.1, $k_{12}^0 + xk_{22}^0 > 0$ on $(a = 0, \mu_1)$ and the assumption that $r < r_0$ on I is contradicted giving $r \equiv r_0$ on (a, μ_1).

To continue the proof, instead of using the more complicated second term of the integrand of (25), use the first term of the right-hand side of (26) or (27) involving $F - F_0 = (q - q_0)Q_1 + (p - p_0)F_1(x)$. An argument, paralleling the preceding one but involving (27), yields that if $\eta(x) = Y_0(x)\xi$ and $q > q_0$ on a subinterval I of (a, μ_1) then $\eta_2(x) \equiv 0$, or if $p > p_0$ on I then $\eta_1(x) + (x - a)\eta_2(x) \equiv 0$ on I, neither of which is compatible with $\eta' = (E_0K_0^{-1})\eta$. Therefore, $\mu_1 = \mu_1^0$ implies that (22) is true and the converse obviously holds, completing the proof of Theorem 4.1.

If equation (1_0) is equation (1) with $q \equiv 0$ or with $p \equiv 0$, that is,

$$(ry'')'' - py = 0 \text{ or } [(ry'')' + qy']' = 0.$$

Then the existence of $\mu_1(a)$ for either of the special cases implies the same for the complete equation (1). Therefore, using the results of §§ 2 and 3:

THEOREM 4.2. *If*

$$\int^{\infty} p(x)(\tilde{I}^2 p)^2 dx = \infty$$

or $\int^{\infty} q(x)dx = \infty$ *then* $\mu_1(b)$ *exists for every* $b \in [a, \infty)$ *and if, in addition,* $\int^{\infty} 1/r = \infty$ *then* $\eta_1(b)$ *exists for every* $b \in [a, \infty)$.

THEOREM 4.3. *If* $\bar{\mu}_1(a)$ *exists for the second-order equation* (15) $(ry')' + qy = 0$ *then* $\mu_1(a)$ *exists for the fourth-order equation* (1) *and* $\mu_1(a) \leqslant \bar{\mu}_1(a)$.

5. Comparison theorems for $\eta_1(a)$. The discussion of this section is an extension of that of the preceding section and consider again the equations (1) and (1_0) and the same ordering of coefficients extended to the first conjugate point:

THEOREM 5.1. *If* $\eta_1^0(a)$ *exists and* $r(x) \leqslant r_0(x)$, $p_0(x) \leqslant p(x)$, $q_0(x) \leqslant q(x)$ *on* $[a, \eta_1^0]$ *then* $\eta_1(a)$ *exists and* $\eta_1(a) \leqslant \eta_1^0(a)$.

Proof. Suppose $\eta_1^0(a)$ exists and $\eta_1(a)$ does not exist on $(a, \eta_1^0]$. In the last section it was established that $H - H_0 \geqslant 0$ on (a, b) for some $b \in (a, \mu_1^0)$ and by (26)–(27) that semi-definiteness extends to the whole interval (a, η_1^0). But by Theorem 3.2:

$$\text{max. e.v. } H(x) \geqslant \text{max. e.v. } H_0(x) \to \infty, \text{ as } x \to \eta_1^0,$$

which contradicts that $H(x) = -\tilde{Y}Y^{-1}$ exists and is continuous on $(a, \eta_1^0]$, thus completing the proof of the theorem.

An immediate corollary is:

THEOREM 5.2. *If* $\eta_1^0(a)$ *exists for either*

$$(ry'')'' - py = 0 \text{ or } [(ry'')' + qy']' = 0$$

then $\eta_1(a)$ *exists for the complete equation* (1) *and* $\eta_1(a) \leqslant \eta_1^0(a)$.

This comparison theorem leads to the following:

THEOREM 5.3. *If the second-order equation* (15) $(r(x)y')' + q(x)y = 0$ *has a non-trivial solution with four zeros on* $[a, \infty)$ *then* $\eta_1(a)$ *exists for the fourth-order equation* (1).

Proof. First suppose that $p(x) \equiv 0$ on $[a, \infty)$. For this special case $\rho(x) \equiv u_1(x)$ and if $z(x) = u'(x)$ then since $u_2'(x) \equiv 0$ and $u_2(a) = 0$:

$$(rz')' + qz = 0, \; z(a) = 0, \; (rz')(a) = 1.$$

Therefore, $z(x)$ has at least four zeros on $[a, \infty)$, the first at $x = a$, and $z'(x)$ has at least three zeros on (a, ∞). Note that $\rho(x) \equiv u_1(x)$, which has the zeros of $z'(x)$ and these guarantee $\mu_1(a)$, $\mu_2(a)$, and $\mu_3(a)$. Theorem 3.4 now gives the existence of $\eta_1(a)$ when $p \equiv 0$. The comparison Theorem 5.2 eliminates the restriction on $p(x)$.

It should be clear that many of the methods of this paper are applicable to self-adjoint equations of higher order and to the matrix equations of Sternberg.

6. Non-existence of $\eta_1(a)$ and disconjugacy. In the last section of their paper **(6)**, Leighton and Nehari gave two changes of variable each of which removes the middle term of

$$[(r(x)y'')' + q(x)y']' - p(x)y = 0. \tag{1}$$

The transformation of the independent variable is repeated here in a slightly more general form.

LEMMA 6.1. *Let I be any subinterval of $[a, \infty)$, $\alpha(x)$ be a function of x such that α and $r\alpha'' \in C'$ with $\alpha' > 0$ on I. Then the change of variable*

$$t = \alpha(x),\ x = \alpha^{-1}(t) = x(t)$$

transforms I into a t-interval I' and (1) into

$$[(R(t)\dot{Y})^{\cdot} + Q(t)\dot{Y}]^{\cdot} - P(t)Y = 0 \text{ on } I', \tag{26}$$

where $Y(t) = y[x(t)]$, $R(t) = r[x(t)]\ (\alpha'[x(t)])^3$,

$$Q(t) = \alpha_2[x(t)],\ \alpha_2 = (r\alpha'')' + q\alpha \text{ and } P(t) = p[x(t)]/\alpha'[x(t)].$$

Note that if $\alpha_2 = 0$ on I then the middle term vanishes and the above lemma reduces to that of Leighton and Nehari:

LEMMA 6.2 **(6,** Theorem 12.1**)**. *If the second-order equation*

$$(r(x)y')' + q(x)y = 0 \text{ on } I = [b, c] \subset [a, \infty) \tag{15}$$

has a positive solution $z(x)$ then the substitution $t = \alpha(x) = \int_b^x z(s)ds$ transforms I into $I' = [0, T]$ and (1) into

$$[R(t)\dot{Y}]^{\cdot\cdot} - P(t)Y = 0 \text{ on } I' \tag{27}$$

where $R(t) = r[x(t)]z^3[x(t)]$ and $P(t) = p[x(t)]/z[x(t)]$.

Proof of Lemma 6.1. As in **(6)**, successive differentiations of $Y(t) = y[x(t)]$ yield $\dot{Y} = y'/\alpha'$ and $\ddot{Y} = (\alpha'y'' - y'\alpha'')/(\alpha')^3$, let $Y_1(t) = R(t)\ddot{Y}(t)$ and $Y_2(t) = \dot{Y}_1 + Q\dot{Y}$, then $\dot{Y}_2 = PY$ (with R, Q, and P as defined in the lemma) and the lemma is proved.

THEOREM 6.1. *If (15) $(ry')' + qy = 0$ is disconjugate (no non-trivial solution has more than one zero) on $[a, b)$, $a < b \leqslant \infty$ and $\eta_1(a)$ does not exist on (a, b) then equation (1) is disconjugate (no non-trivial solution has more than three zeros on $[a, b)$.*

Proof. Let $c \in (a, b)$, then, by a well-known property of disconjugate equations, there exists a solution $z(x)$ of (15) which is positive on the closed finite interval $[a, c]$. As in Lemma 6.2, the substitution $t = \alpha(x) = \int_a^x z$

transforms $[a, c]$ into $[0, T]$ and (1) into (27). Also, $\eta_1(0)$ does not exist with respect to equations (27) on $(0, T]$ and, by **(6)**, no solution of (27) has more than three zeros on $[0, T]$. Therefore, no solution of (1) has more than three zeros on $[a, c]$ and, hence, on $[a, b)$.

THEOREM 6.2. *If $\eta_1(a)$ does not exist, $p(x) + q(x) \not\equiv 0$ for large x, and $\int^\infty 1/r = \infty$ then equation* (1) *is disconjugate.*

Proof. By Theorem 3.1, the non-existence of $\eta_1(a)$ implies the non-existence of $\mu_1(a)$ and Theorem 4.1 implies that $\mu_1(a)$ does not exist with respect to (1) with $p \equiv 0$. Hence, by Theorem 2.2, the second-order equation (15) is a disconjugate and Theorem 6.1 guarantees that equation (1) is disconjugate and the proof is complete.

If $\int^\infty 1/r < \infty$ or $(ry')' + qy = 0$ is oscillatory, it is not known to the author whether the non-existence of $\eta_1(a)$ implies disconjugacy of (1). Of course, by **(6**, Part II**)**, if $p(x) < 0$ and $q \equiv 0$ then neither $\mu_1(a)$ nor $\eta_1(a)$ exists, but it should be recalled that in many cases the fundamental solution $v(x)$ has a simple zero following its initial triple zero.

REFERENCES

1. J. H. Barrett, *A Prüfer transformation for matrix differential equations*, Proc. Amer. Math. Soc., *8* (1957), 510–518.
2. ——— *Disconjugacy of second-order linear differential equations with non-negative coefficients*, Proc. Amer. Math. Soc., *10* (1959), 552–561.
3. ——— *Disconjugacy of a self-adjoint differential equation of the fourth order*, OOR Technical Report (University of Utah, December 1959), chapter I, 1–24, Pacific J. Math., *11* (1961), 25–37.
4. ——— *Systems-disconjugacy of a fourth-order differential equation*, Technical Summary Report No. 135, Mathematics Research Center, United States Army (1960), Proc. Amer. Math. Soc., *12* (1961), 205–213.
5. H. C. Howard, *Oscillation criteria for fourth-order linear differential equations*, Trans. Amer. Math. Soc., *96* (1960), 296–311.
6. W. Leighton and Z. Nehari, *On the oscillation of solutions of self-adjoint linear differential equations of the fourth order*, Trans. Amer. Math. Soc., *89* (1958), 325–377.
7. W. T. Reid, *A matrix differential equation of the Riccati type*, Amer. J. Math., *68* (1946) 237–246.
8. R. L. Sternberg, *Variational methods and nonoscillation theorems for systems of differential equations*, Duke Math. J., *19* (1952) 311–322.
9. R. L. Sternberg and H. Kaufman, *A two-point boundary problem for ordinary self-adjoint differential equations of even order*, Duke Math. J., *20* (1953) 527–531.
10. R. L. Sternberg and H. M. Sternberg, *A two-point boundary problem for ordinary self-adjoint differential equations of the fourth order*, Can. J. Math., *6* (1954) 416–419.

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GENERAL RADICALS THAT COINCIDE WITH THE CLASSICAL RADICAL ON RINGS WITH D.C.C.

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General radical theories were obtained by Amitsur **(1; 2; 3)** and Kurosh **(6)**. Following Kurosh we say that a property $\mathfrak{S}$ of rings is a radical property if:

(a) *Every homomorphic image of an $\mathfrak{S}$-ring is an $\mathfrak{S}$-ring;*

(b) *Every ring R contains an $\mathfrak{S}$-ideal S which contains every other $\mathfrak{S}$-ideal of R;*

(c) *The factor ring R/S is $\mathfrak{S}$-semi-simple (that is, has no non-zero $\mathfrak{S}$-ideals).*

The property $\mathfrak{N}$ of being nil is a radical property and for rings with D.C.C. (the descending chain condition on left ideals) this becomes the so-called Classical Radical. Nilpotency is not a radical property for the union of all the nilpotent ideals of a general ring need not be nilpotent. However, for rings with D.C.C. all nil radicals are nilpotent.

The question we are concerned with is which general radical properties coincide with $\mathfrak{N}$ on rings with D.C.C. If $\mathfrak{S}$ and $\mathfrak{T}$ are two radical properties we say $\mathfrak{S} \leqq \mathfrak{T}$ if every $\mathfrak{S}$-radical ring is also $\mathfrak{T}$-radical, and if we work with the class of all (associative) rings this is equivalent to the statement that for any ring R, the $\mathfrak{S}$-radical of R is contained in the $\mathfrak{T}$-radical of R. We say $\mathfrak{S} = \mathfrak{T}$ if a ring is $\mathfrak{S}$-radical if and only if it is $\mathfrak{T}$-radical, or if, for every ring R, its $\mathfrak{S}$-radical equals its $\mathfrak{T}$-radical. However, if we consider only rings with D.C.C. these statements are not equivalent for an ideal of a ring with D.C.C. may not have D.C.C. itself. Thus it is possible to have properties $\mathfrak{S}$ and $\mathfrak{T}$ such that a ring with D.C.C. is $\mathfrak{S}$-radical if and only if it is $\mathfrak{T}$-radical, but there exist rings R with D.C.C. whose $\mathfrak{S}$-radicals are smaller than their $\mathfrak{T}$-radicals. We seek general radical properties $\mathfrak{Q}$ which coincide with $\mathfrak{N}$ on rings with D.C.C. in the strong sense that for any ring R with D.C.C., its $\mathfrak{Q}$ radical equals its $\mathfrak{N}$ radical. To this end we shall use Kurosh's upper and lower radical constructions.

Given any set of rings P, the *lower radical property* determined by P is defined as follows:

A ring is of first degree over P if it is a homomorphic image of some ring in P or if it is zero (this is to complete the definition in case P is vacuous). We say a ring R is of degree $\beta > 1$ over P if every non-zero homomorphic

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image of R contains a non-zero ideal which is a ring of degree $\beta - 1$ over P. If β is a limit ordinal we say a ring is of degree β over P if it is of some degree $\alpha < \beta$, over P. Then we consider all rings of any degree over P and we say a ring is radical if it is of some degree over P. This yields a radical property for which all rings in P are radical and it is less than or equal to any other radical property for which all rings in P are radical.

Given any set of rings Q with the following property:

(d) *Every non-zero ideal of a ring of Q can be homomorphically mapped onto some non-zero ring of Q;*

the *upper radical property* determined by Q is defined as follows:

We consider the class $\bar{Q}$, the set of all rings R such that every non-zero ideal of R can be homomorphically mapped onto some non-zero ring of Q. We then say a ring is radical if it cannot be homomorphically mapped onto a non-zero ring of $\bar{Q}$. This yields a radical property for which all rings in Q are semi-simple and it is bigger than or equal to any other radical property for which all rings in Q are semi-simple.

We make the following definitions:

$\mathfrak{L}$ = *the lower radical property determined by all the zero simple rings.*

$\mathfrak{D}$ = *the lower radical property determined by all nilpotent rings which are nil radicals of rings with D.C.C.*

$\mathfrak{B}$ = *the lower radical property determined by all nilpotent rings.*

$\mathfrak{N}$ = *the lower radical property determined by all nil rings.*

$\mathfrak{U}$ = *the upper radical property determined by all finite dimensional total matric rings over division rings* (*since this class consists only of simple rings it clearly has property D.*)

It is clear then that

$$\mathfrak{L} < \mathfrak{D} < \mathfrak{B} < \mathfrak{N} < \mathfrak{U}.$$

If $\mathfrak{Q}$ is any radical property that coincides with $\mathfrak{N}$ on rings with D.C.C. then all zero simple rings are $\mathfrak{Q}$-radical, for zero simple rings have D.C.C. and are $\mathfrak{N}$-radical. Therefore $\mathfrak{L} < \mathfrak{Q}$. On the other hand, all finite dimensional total matric rings over division rings have D.C.C., are nil semi-simple and are therefore $\mathfrak{Q}$ semi-simple. Thus $\mathfrak{Q} < \mathfrak{U}$. Consequently $\mathfrak{L} < \mathfrak{Q} < \mathfrak{U}$. However, every nilpotent ring is of course $\mathfrak{N}$-radical and if it is a nil radical of a ring with D.C.C. then it must also be $\mathfrak{Q}$ radical. Therefore $\mathfrak{D} < \mathfrak{Q} < \mathfrak{U}$.

THEOREM 1. *A general radical property $\mathfrak{Q}$ coincides with the nil radical $\mathfrak{N}$ on rings with D.C.C. if and only if $\mathfrak{D} < \mathfrak{Q} < \mathfrak{U}$.*

Proof. We have already proved half of this theorem. To prove the other half it is sufficient to show that both $\mathfrak{D}$ and $\mathfrak{U}$ coincide with $\mathfrak{N}$ on rings with

D.C.C. To see that $\mathfrak{N}$ and $\mathfrak{U}$ coincide let R be any ring with D.C.C. Let N be its $\mathfrak{N}$-radical and U be its $\mathfrak{U}$-radical. $N \subseteq U$. We consider R/N which is well known to be a finite direct sum of D_i's, where the D_i are finite dimensional total matrix rings over division rings. Now U/N is an ideal of R/N and must then be a finite direct sum of the D_i's that it contains. Then U can be homomorphically mapped, via U/N, onto one of the D_i which is $\mathfrak{U}$-semi-simple. However, U is $\mathfrak{U}$-radical and so is every homomorphic image of U and the only ring that is both radical and semi-simple is the ring consisting only of zero. Thus U/N must be zero, $U = N$.

To see that $\mathfrak{N}$ and $\mathfrak{D}$ coincide again let R be any ring with D.C.C. Let N be its $\mathfrak{N}$-radical and D its $\mathfrak{D}$-radical. $D \subseteq N$. Now N is nilpotent and it is a nil radical of a ring with D.C.C. and therefore N is $\mathfrak{D}$-radical, $N = D$. Q.E.D.

Kurosh makes the statement that a general radical property $\mathfrak{Q}$ coincides with the nil radical $\mathfrak{N}$ on rings with D.C.C. if and only if $\mathfrak{L} \leqq \mathfrak{Q} \leqq \mathfrak{U}$. However, he overlooked the fact that though this is true in the weak sense that every $\mathfrak{Q}$-radical ring with D.C.C. is $\mathfrak{L}$-radical, this is false in the strong sense as the following example shows:

Let A be the set of all $\alpha x + \beta e$ where α and β are rational numbers and where $x^2 = 0$, $e^2 = e$, $ex = xe = x$. This is a commutative ring which is a two-dimensional vector space over the rationals. The only non-zero proper ideal of A is $N = \{\alpha x\}$. Clearly N is the nil radical of A and A has D.C.C. We want to show that A is $\mathfrak{L}$-semi-simple. The $\mathfrak{L}$-radical of A is contained in N and thus it remains to show that N is $\mathfrak{L}$-semi-simple. Assume then that N contains some $\mathfrak{L}$-ideals, each of them being of some degree over the class of all zero simple rings. Let γ be the minimal ordinal such that N has an ideal I which is of degree γ. Clearly γ is not a limit ordinal. Since I is of degree γ, every non-zero homomorphic image of I must contain a non-zero ideal of degree $\gamma - 1$ and in particular I itself must contain a non-zero ideal J of degree $\gamma - 1$. However, since $N^2 = 0$, $JN = 0$ and J is therefore an ideal of N. Thus N contains a non-zero ideal of degree $\gamma - 1$, which contradicts the minimality of γ, unless $\gamma = 1$. Thus if N contains any $\mathfrak{L}$-ideals it must contain one of degree 1. However, any homomorphic image of a zero simple ring is a zero simple ring and thus the only rings of degree 1 are the zero simple rings themselves. However, any non-zero ideal of N is merely an additive subgroup and contains at least the infinite cyclic additive group generated by a non-zero element, and thus cannot be simple. Therefore N has no $\mathfrak{L}$-ideals and N is $\mathfrak{L}$-semi-simple.

Thus A is a ring with D.C.C. whose $\mathfrak{N}$-radical is not equal to its $\mathfrak{L}$-radical. This proves also that $\mathfrak{L} \lneqq \mathfrak{D}$.

What is true is that every $\mathfrak{N}$-radical ring R with D.C.C. is also $\mathfrak{L}$-radical. For if R is nilpotent, $R^m = 0 \neq R^{m-1}$. By D.C.C. R^{m-1} contains a minimal ideal I of R. If J is any ideal of I it is also an ideal of R for $JR \subseteq R^{m-1}R = 0$. Thus I is a zero simple ring. Thus R contains a zero simple ring. Similarly,

every homomorphic image of R contains a zero simple ring and therefore R is of degree 2 over the zero simple rings and thus R is $\mathfrak{L}$-radical.

Of course in the example above, though N is contained in a ring with D.C.C., N itself does not have D.C.C. Curiously every homomorphic image of that N, which is not isomorphic to N, is $\mathfrak{L}$-radical.

It is clear that, in general, $\mathfrak{U} \neq \mathfrak{R}$ for the set of all rational numbers of the form $2m/(2n+1)$ is a Jacobson radical ring which is clearly $\mathfrak{U}$-radical, but is $\mathfrak{R}$-semi-simple. Also $\mathfrak{B} \neq \mathfrak{N}$, for $\mathfrak{N}$ is the Baer upper radical and $\mathfrak{B}$ is the Baer lower radical and Baer **(5, § 2)** has given an example where they are different. To see that $\mathfrak{B}$ is the Baer lower radical we first point out that $\mathfrak{B}$ is identical with the lower radical property determined by the zero ring on an infinite cyclic additive group W. Clearly if all nilpotent rings are radical then in particular W is radical. On the other hand, every nilpotent ring contains an ideal which is a zero ring on a cyclic additive group and this is a homomorphic image of W. Thus every nilpotent ring is of degree 2 over W and thus if W is radical so are all nilpotent rings. Kurosh has pointed out that the lower radical determined by W is precisely Baer's lower radical.

To see that all five radical properties are different in general, that is,

$$\mathfrak{L} \subsetneq \mathfrak{D} \subsetneq \mathfrak{B} \subsetneq \mathfrak{N} \subsetneq \mathfrak{U},$$

we must finally show that $\mathfrak{D} \neq \mathfrak{B}$.

THEOREM 2. *The radical property* $\mathfrak{D} \subsetneq \mathfrak{B}$, *the Baer lower radical.*

Proof. We know that $\mathfrak{D} \leqslant \mathfrak{B}$ and the question as to their equality will be settled if we show that W, the zero ring on an infinite cyclic additive group, is $\mathfrak{D}$-semi-simple.

Suppose then that W contains some $\mathfrak{D}$-ideals. However, every non-zero ideal of W is isomorphic to W and thus if W has a non-zero $\mathfrak{D}$-ideal, it must be $\mathfrak{D}$-radical itself. Let α be the minimal ordinal such that W is of degree α over the class of all nilpotent rings which are nil radicals of rings with D.C.C. Then clearly α is not a limit ordinal. Every non-zero homomorphic image of W then contains a non-zero ideal of degree $\alpha - 1$ and in particular W contains such an ideal and therefore W itself is of degree $\alpha - 1$ which contradicts the minimality of α, unless $\alpha = 1$. Then W is a homomorphic image of a nil radical of a ring with D.C.C.

Let R be a ring with D.C.C., let A be its nil radical and let H be an ideal of A such that $A/H \cong W$. Then $A^2 \subseteq H$. We consider then the ring R/A^2. It also has D.C.C. Its nil radical is known to be A/A^2 and this can be homomorphically mapped onto W for $A/A^2/H/A^2 \cong A/H \cong W$. Thus we may assume without loss of generality that $A^2 = 0$.

Every element of A is of the form $mx + h$ where m is an integer, h is in H, and where x is a representative of the generator of the infinite cyclic additive group A/H. Note that if mx is in H then m must be zero.

Consider the sequence of left ideals of R:

$$R > Rx > R\cdot 2x > R\cdot 2^2x > \ldots > R\cdot 2^nx > \ldots .$$

None of these can be zero for if $R\cdot 2^nx = 0$ then

$$\{2^nx\} > \{2^{n+1}x\} > \ldots > \{2^{n+r}x\} > \ldots$$

where $\{2^{n+r}x\}$ is the additive group generated by $2^{n+r}x$, is a properly descending chain of non-zero left ideals of R, which contradicts D.C.C.

Again by D.C.C. there must exist an integer n such that

$$R\cdot 2^nx = R\cdot 2^{n+1}x.$$

Thus for every element f in R there must exist an element g in R such that $f\cdot 2^nx = g\cdot 2^{n+1}x$.

We note that $R \neq A$ else A has D.C.C. and therefore $A/H \cong W$ has D.C.C. which is impossible. Therefore R contains an idempotent e such that every element a of R is: $a = ae + (a - ae)$ where $a - ae$ is in A (**4**, pp. 17–19). Thus $ax = aex$, since $A^2 = 0$. Then $a(x - ex) = 0$ for every a, $R(x - ex) = 0$.

If $ex = x + h$ then there must exist an element b in R such that $e\cdot 2^nx = b\cdot 2^{n+1}x$. However, bx is in A and therefore $bx = mx + h'$, where m is an integer. Then $e\cdot 2^nx = 2^nx + 2^nh = b\cdot 2^{n+1}x = 2^{n+1}mx + 2^{n+1}h'$. Therefore $x(2^n - m2^{n+1}) = 2^{n+1}h' - 2^nh$ which is in H. Thus $2^n - m2^{n+1} = 0$ which is impossible.

On the other hand, if $ex \neq x + h$ then $ex = qx + h_1$ where $q \neq 1$ and $x - ex \neq 0$. Then $\{x - ex\} > \{2(x - ex)\} > \{2^2(x - ex)\} > \ldots > \{2^n(x - ex)\} > \ldots$ is a descending chain of left ideals of R; where again $\{2^n(x-ex)\}$ is the additive group generated by $2^n(x - ex)$. Each is non-zero for if $2^n(x - ex) = 0$ then $2^n(x - qx - h_1) = 2^n(1 - q)x - 2^nh_1 = 0$. Thus $2^n(1 - q)x$ is in H which is impossible unless $q = 1$.

This is a properly descending chain for if $\{2^n(x - ex)\} = \{2^{n+1}(x - ex)\}$, then there must exist an integer k such that

$$2^n(x - ex) = k2^{n+1}(x - ex).$$

$$\therefore 2^n(x - qx - h_1) = k\cdot 2^{n+1}(x - qx - h_1).$$

$$\therefore x[2^n(1 - q) - k2^{n+1}(1 - q)] = h_1(2^n - k2^{n+1})$$

which is in H.

$$\therefore 2^n(1 - q)(1 - 2k) = 0$$

which is impossible for an integer k.

Thus in either case we have a contradiction. Therefore W is $\mathfrak{D}$-semi-simple and the theorem is proved.

References

1. S. A. Amitsur, *A general theory of radicals*, Amer. J. Math., *74* (1952), 774–786.
2. ——— *A general theory of radicals II*, Amer. J. Math., *76* (1954), 100–125.
3. ——— *A general theory of radicals III*, Amer. J. Math., *76* (1954), 126–136.
4. Artin, Nesbitt, and Thrall, *Rings with minimum conditions*. University of Michigan, publ. no. 1 (1946).
5. R. Baer, *Radical ideals*, Amer. J. Math. *65* (1943), 537–567.
6. A. G. Kurosh, *Radicals of rings and algebras*, Math. Sbornik. V, *33* (75) (1953).

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THE ANALYTICITY OF KERNELS

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Let V be a paracompact real analytic manifold of dimension $n \geq 1$. Following the terminology of the theory of distributions of Schwartz **(4)**, $\mathfrak{D}(V)$ is the linear space of infinitely differentiable functions with compact support in V with the appropriate inductive limit topology, $\mathfrak{E}(V)$ is the Frechet space of infinitely differentiable functions on V, $\mathfrak{D}'(V)$ is the dual space of $\mathfrak{D}(V)$ consisting of the distributions on V, $\mathfrak{E}'(V)$ the dual space of $\mathfrak{E}(V)$ consisting of the distributions with compact support on V. Let $\mathfrak{A}(V)$ be the linear space of real analytic functions on V.

It is our purpose in the present paper to study the nature of the distribution kernels on $V \times V$, in the sense of Schwartz **(5)**, which are analytically very regular. Such kernels, which we shall refer to more briefly as analytic kernels, are obtained by the Schwartz kernel theorem from analytic mappings. The latter are defined as follows: Let L be a continuous linear mapping of $\mathfrak{D}(V)$ into $\mathfrak{E}(V)$. Its transpose (or adjoint) ${}^tL$ is then defined as a continuous linear mapping of $\mathfrak{E}'(V)$ into $\mathfrak{D}'(V)$. Then L is said to be analytic (or bi-analytic) if the following two conditions hold:

(i) ${}^tL$ maps $\mathfrak{D}(V)$ into $\mathfrak{E}(V)$. (It then follows by the closed graph theorem that ${}^tL$ maps $\mathfrak{D}(V)$ continuously into $\mathfrak{E}(V)$, so that by the reflexivity of the space $\mathfrak{E}(V)$ and $\mathfrak{D}(V)$, it follows that ${}^t({}^tL) = L$ maps $\mathfrak{E}'(V)$ continuously into $\mathfrak{D}'(V)$.)

(ii) If ψ lies in $\mathfrak{E}'(V)$ and is analytic on an open subset G of V, then $L\psi$ and ${}^tL\psi$ are both analytic on G.

Associated with each such mapping L by the Schwartz kernel theorem **(5)**, is a distribution $k_{x,y}$ on $V \times V$, infinitely differentiable off the diagonal, such that figuratively

$$(L\psi)(x) = \int k_{x,y}\psi(y)dy,$$

for all ψ in $\mathfrak{D}(V)$, or more precisely, using the pairing $\langle f, \Phi \rangle$ between a distribution f on $V \times V$ and $\Phi \in \mathfrak{D}(V \times V)$ (and similarly on V),

$$\langle L\psi, \psi_1 \rangle = \langle k_{x,y}, \psi(y)\psi_1(x) \rangle,$$

for all ψ, ψ_1 in $\mathfrak{D}(V)$.

Our principal result is the following:

THEOREM. *The kernel $k_{x,y}$ of an analytic mapping L is analytic on the complement of the diagonal in $V \times V$.*

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The corresponding result with analyticity replaced by separate analyticity (that is, analyticity in each of x and y separately with the other held fixed) was obtained by de Barros-Neto in **(1)**. An essential tool in the proof of the above Theorem is the criterion for analyticity on $V \times V$ in terms of the analytic properties of a function in the separate variables given by Browder in **(2)**.

Obviously our theorem has important applications to the study of the structure of the fundamental solutions and other kernels obtained from boundary value problems for partial differential operators. We shall discuss these applications elsewhere.

In § 2, we consider some generalizations to general kernels.

The writers are indebted to Professor L. Nachbin for some stimulating discussions of the questions dealt with here.

1. We shall assume as the basis of a great deal of our discussion the theory of topological vector spaces as exposed by Grothendieck in **(3)**. By a standard procedure, we may imbed V in a complex analytic manifold $\tilde{V}$ of dimension n such that at each point x^0 of V, there exists a complex analytic co-ordinate system $\{z_1, \ldots, z_n\}$ in a neighbourhood U of x^0 in $\tilde{V}$ such that $U \cap V \subset \{z: \mathrm{Im}(z_j) = 0 \text{ for } 1 \leqslant j \leqslant n\}$, while $x_j = \mathrm{Re}(z_j)$ are the co-ordinates of a co-ordinate patch in V at x^0. We shall refer to $\tilde{V}$, which we shall fix, as the standard complexification of V.

If u is a function from $\mathfrak{A}(V)$, there exists a neighbourhood U of V in $\tilde{V}$ such that u may be extended to a holomorphic function on U. We shall denote the linear space of holomorphic functions on U by $\mathfrak{H}(U)$, topologizing the latter space by the usual topology of uniform convergence on compact subsets of U. We then define the topology on $\mathfrak{A}(V)$ as the inductive limit of the topologies induced by the mappings of $\mathfrak{H}(U)$ into $\mathfrak{A}(V)$ obtained by restricting functions from $\mathfrak{H}(U)$ to V **(3**, chapter IV, pp. 255–256**)**, where U runs over the neighbourhoods of V in $\tilde{V}$. Similarly, if A is a compact subset of V, $\mathfrak{A}(A)$ will denote the space of equivalence classes of functions analytic on a neighbourhood of A in $\tilde{V}$, and topologized as the inductive limit of $\mathfrak{H}(U_k)$ where U_k runs through a fundamental sequence of neighbourhoods of A in $\tilde{V}$. It follows that $\mathfrak{A}(A)$ is the inductive limit of a sequence of Frechet spaces. Grothendieck has shown **(3**, Theorem 1, Corollary 2, pp. 268–270**)** that if every bounded closed subset of a generalized $(\mathfrak{LF})$-space E is complete and if $\{E_j\}$ is a sequence of definition for the space E, then each bounded subset B of E is contained and bounded in one of the spaces E_j. By **(3**, 4(6), p. 315**)**, for a compact set A in V, $\mathfrak{A}(A)$ is complete, and hence quasi-complete (that is, bounded closed subsets are complete in the induced uniformity). It follows then that we have:

LEMMA 1. *For each compact subset A of V, if B is a bounded subset of $\mathfrak{A}(A)$, then each function f in B can be extended to a fixed neighbourhood U of A in $\tilde{V}$*

(depending of course upon B), and there exists a constant M such that for the extended function $f(z)$,

$$|f(z)| \leqslant M$$

for z in U.

Proof. The proof follows from the above remarks together with the familiar form of the bounded subsets in $\mathfrak{H}(U_1)$, where U_1 is a neighbourhood containing the compact $\bar{U}$ in its interior.

LEMMA 2. *A necessary and sufficient condition that a subset B of $\mathfrak{A}(A)$ should be bounded in $\mathfrak{A}(A)$, A compact, is that there exists a finite family of co-ordinate patches on A such that in each of the co-ordinate patches there exists a constant C_0 such that*

$$\left|\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\left(\frac{\partial}{\partial x_2}\right)^{\alpha_2}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} f(x)\right| \leqslant C_0^{\Sigma_j \alpha_j}(\pi_j \alpha_j!)$$

Proof. This is an easy consequence of the Cauchy integral formula for polycylinders as far as necessity is concerned, and of an obvious majorization of the power series of f as far as sufficiency is concerned.

By another result of Grothendieck **(3**, first Corollary on p. 36**)**, it follows that if E is a Frechet space which is also a Schwartz space, then its dual E' is a complete bornological space. By **(3**, Proposition 5, p. 336**)**, for every open subset G of the separable manifold V, $\mathfrak{E}(G)$ is a Schwartz space and obviously a Frechet space. Hence $\mathfrak{E}'(G)$ is a complete bornological space. By **(3**, Theorem 2, p. 271**)**, every closed mapping of a complete bornological space E (hence strictly bornological) into a generalized $(\mathfrak{LF})$-space F is continuous. Applying this result to the spaces $E = \mathfrak{E}'(G)$, $F = \mathfrak{A}(A)$ with A compact, we obtain:

LEMMA 3. *Let T be a closed linear mapping of $\mathfrak{E}'(G)$ into $\mathfrak{A}(A)$ with G open in V, A compact. Then T is continuous. In particular, this is true if T is the restriction of a continuous mapping of $\mathfrak{E}'(G)$ into $\mathfrak{D}'(V)$.**

Proof. The first assertion of Lemma 3 was established in the remarks preceding the statement of the Lemma. For the second assertion, it suffices to show that if T is the restriction of a continuous mapping of $\mathfrak{E}'(G)$ into $\mathfrak{D}'(V)$, then T is closed as a mapping of $\mathfrak{E}'(G)$ into $\mathfrak{A}(A)$. However, the topology of $\mathfrak{A}(A)$ is consistent with the induced topology from $\mathfrak{D}'(V)$, so that the last conclusion follows immediately.

LEMMA 4. *Let f be a distribution on $V \times V$ such that f is separately analytic off the diagonal in $V \times V$ (that is, $f(x,.)$ is analytic for fixed x in y for $y \neq x$,*

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*By this we mean precisely the following: There exists a continuous mapping T_1 of $\mathfrak{E}'(G)$ into $\mathfrak{D}'(V)$ such that for all u, T_1u is analytic on a neighbourhood of A and Tu is the equivalence class of T_1u taken as an element of $\mathfrak{A}(A)$.

while $f(.,y)$ is analytic for fixed y for $x \neq y$). Suppose the mappings Φ_1, and Φ_2, with

$$\Phi_1(x) = f(x,.), \qquad \Phi_2(y) = f(.,y)$$

are continuous mappings of an open set G_1 into $\mathfrak{A}(G_2)$ for another open set G_2 (G_2 into $\mathfrak{A}(G_1)$) in V, $G_1 \cap G_2 = \phi$. Then f is analytic on $G_1 \times G_2$.

Proof. It suffices to consider compact subsets A_1 and A_2 of G_1 and G_2, respectively. Since Φ_1 and Φ_2 are continuous, $\Phi_1(A_1)$ is bounded in $\mathfrak{A}(A_2)$ and $\Phi_2(A_2)$ is bounded in $\mathfrak{A}(A_1)$. Hence, by Lemma 1, there exist neighbourhoods U_1 of A_1 in $\bar{V}$ and U_2 of A_2 in $\bar{V}$ such that $f(x,.)$ may be extended for fixed x in A_1 to a function $f(x,z)$ holomorphic for z in A_2 with $|f(x,z)| \leqslant M$, while $f(.,y)$ may be extended to a holomorphic function $f(z,y)$ on U_1 for fixed y in A_2 with $|f(z,y)| \leqslant M$ for all z in U_1, y in A_2. However, these are precisely the hypotheses of Theorem 1 of Browder **(2)** in order that f should be analytic on $G_1 \times G_2$, and thus the Lemma is established.

LEMMA 5. *Let $k_{x,y}$ be a distribution on $V \times V$, infinitely differentiable off the diagonal. Let G_1 and G_2 be two open subsets of V with disjoint compact closures. Then for each x in G_1, y in G_2,*

$$k_{x,y} = L(\delta_y)(x),$$

$$k_{x,y} = {}^tL(\delta_x)(y).$$

Proof. Since $k_{x,y}$ is infinitely differentiable off the diagonal, k is infinitely differentiable on $G_1 \times G_2$. It follows that

$$L(\delta_y)(x) = \langle k_{x,y_1}, \delta_y(y_1) \rangle = k_{x,y}$$

The other equation follows similarly.

Proof of the Theorem. Let G_1 and G_2 be any two open subsets of V with disjoint compact closures. It suffices to prove that $k_{x,y}$ is analytic on $G_1 \times G_2$. Let $\mathfrak{E}'(G_1)$ be imbedded in $\mathfrak{E}'(V)$ in the natural way. Then for each u in $\mathfrak{E}'(G_1)$, u realized as an element of $\mathfrak{E}'(V)$ is analytic (in fact identically zero) on G_2. Thus ${}^tLu$ restricted to G_2 lies in $\mathfrak{A}(G_2)$. Let T be this restriction. By Lemma 3, T is a continuous mapping of $\mathfrak{E}'(G_1)$ into $\mathfrak{A}(G_2)$. Consider

$$\Phi_1(x) = ({}^tL)(\delta_x).$$

Φ_1 is a continuous mapping of G_1 in $\mathfrak{A}(G_2)$. Similarly, if Φ_2 is given by

$$\Phi_2(y) = L(\delta_y),$$

Φ_2 is a continuous map of G_2 into $\mathfrak{A}(G_1)$. But $k_{x,y} = \Phi_1(x)(y) = \Phi_2(y)(x)$, by Lemma 5.

Hence, by Lemma 4, $k_{x,y}$ is analytic on $G_1 \times G_2$, and the proof of Theorem 1 is complete.

The above proof of Theorem 1 may be applied with purely formal changes to establish the following:

THEOREM 2. *Let V and W be two paracompact real analytic manifolds, L a mapping of $\mathcal{E}'(W)$ into $\mathcal{E}(V)$, ${}^tL$ the dual map of $\mathcal{E}'(V)$ into $\mathcal{E}(W)$. Suppose that $L(\mathcal{E}'(W))$ and ${}^tL(\mathcal{E}'(V))$ are composed of real analytic functions. Then the kernel $k_{x,y}$ of L is analytic in $V \times W$.*

2. We state without going into detail a simple generalization of the results of § 1 to general kernels, which may be used in particular for the integral kernels associated with hyperbolic boundary value problems. We shall return to this question in another place.

If A is an open set of the real analytic manifold, V, we designate by $\mathcal{E}_A'(V)$ the subspace of $\mathcal{E}'(V)$ of distributions which are analytic on some neighbourhood of A in V.

We shall consider a distribution kernel $k_{x,y}$ on $V \times W$, L and ${}^tL$ the associated mappings, with L mapping $\mathcal{D}(W)$ into $\mathcal{D}'(V)$, ${}^tL$ mapping $\mathcal{D}(V)$ into $\mathcal{D}'(W)$.

DEFINITION. *Let y be a point of W. The closed subset A of V is said to be a domain of dependence for y with respect to ${}^tL$ if there exists a neighbourhood U of y in W such that ${}^tL$ can be extended to a continuous mapping of $\mathcal{E}_A'(V)$ into $\mathcal{D}'(U)$ (restricting the values of ${}^tL$ to U), such that the image ${}^tL(\mathcal{E}_A'(V))$ is composed of functions analytic in U. A similar definition may be given for a domain of dependence A' for x in V with respect to L.*

DEFINITION. *The point (x, y) in $V \times W$ is said to be regular with respect to L provided that there exists a domain of dependence A for y with respect to ${}^tL$ which does not contain x and a domain of dependence A' for x with respect to L which does not contain y.*

THEOREM 3. *Let $k_{x,y}$ be a distribution kernel, R the set of points in $V \times W$ which are regular with respect to $k_{x,y}$ (that is, regular with respect to the corresponding mapping L of which $k_{x,y}$ is the kernel). Then R is an open subset of $V \times W$, and $k_{x,y}$ is analytic on R.*

The proof of Theorem 3 is a simple variant of the proof of Theorem 1.

REFERENCES

1. J. de Barros-Neto, Thesis (University of São Paulo, 1960).
2. F. E. Browder, *Real analytic functions on product spaces and separate analyticity*, Can. J. Math., *13* (1961), 650–656.
3. A. Grothendieck, *Espaces vectoriels topologiques* (São Paulo, 1954).
4. L. Schwartz, *Théorie des distributions* (Paris, 1950–1).
5. ——— *Théorie des noyaux*, Cambridge International Mathematical Congress (1950).

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REAL ANALYTIC FUNCTIONS ON PRODUCT SPACES AND SEPARATE ANALYTICITY

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Let f be a function on the product space $V \times W$, where V and W are analytic manifolds, both either real or complex. The function f is said to be analytic (or bi-analytic) on $V \times W$ if it is analytic in the analytic structure induced on $V \times W$ by the corresponding structures on V and W. The function f is said to be separately analytic on $V \times W$ if, for each x in V, the function $f(x,\cdot)$ is analytic on W while, for each y in W, the function $f(\cdot,y)$ is analytic on V. In the case of complex analytic manifolds, the classical theorem of Hartogs **(3**, chapter VII**)** states that the two notions of analyticity and separate analyticity are equivalent. For real analytic manifolds, it is known that such an equivalence does not hold, even if one adds the additional hypothesis that f is infinitely differentiable on $V \times W$.

It is the purpose of the present paper to establish a positive criterion for bi-analyticity in terms of the analytic properties of f in its separate variables. There are several equivalent forms in which we may state this criterion. The most direct of these is the following:

(A) *Consider V and W as imbedded in $\tilde{V}$ and $\tilde{W}$, their complexifications which are complex analytic manifolds.* Let K_1 and K_2 be compact subsets of V and W, respectively. Then there exists a neighbourhood U_1 of K_1 in $\tilde{V}$ and a neighbourhood U_2 of K_2 in $\tilde{W}$ and a constant M (depending on K_1 and K_2) such that for each x in K_1, the function $f(x,\cdot)$ on K_2 may be extended to a complex analytic function $f(x,z)$ on U_2 for which $|f(x,z)| \leqslant M$, $z \in U_2$, while similarly for each y in K_2, the function $f(\cdot,y)$ on K_1 may be extended to a complex analytic function $f(z,y)$ for z in U_1 for which $|f(z,y)| \leqslant M$, $z \in U_1$.*

Our study of this problem arose from a question raised by de Barros-Neto in connection with his investigation **(1)** of the structure of distribution kernels (in the sense of Schwartz) which are analytically very regular; the application of our result (Theorem 1 below) to such kernels is carried out in a joint paper by Barros and the author which appears immediately after the present paper in the same issue of this journal.

The criterion (A) is of local character, as follows easily from the elementary properties of holomorphic functions. We therefore may give another equivalent form if we assume that V and W each lie in a single co-ordinate patch, and therefore without loss of generality, that V and W are E^n and $E^{n'}$, respectively.

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*As defined, for example, by Bruhat and Whitney, Comm. Helv., *33* (1959), 132–60.

(B) There exists a constant C_0 such that for all x in V,

$$\left|\left(\frac{\partial}{\partial y_1}\right)^{\alpha_1}\left(\frac{\partial}{\partial y_2}\right)^{\alpha_2}\cdots\left(\frac{\partial}{\partial y_n}\right)^{\alpha_n} f(x, y)\right| \leqslant C_0^{(\Sigma\alpha_j)} \cdot \left(\sum \alpha_j\right)!$$

and for all y in W,

$$\left|\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\left(\frac{\partial}{\partial x_1}\right)^{\alpha_2}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} f(x, y)\right| \leqslant C_0^{(\Sigma\alpha_j)} \cdot \left(\sum \alpha_j\right)!$$

THEOREM 1. *Let $f(x, y)$ be a Borel measurable function on the product $V \times W$ of two real analytical manifolds V and W such that f is separately analytic on $V \times W$ and satisfies condition* (A) *above. Then f is analytic on $V \times W$.*

Proof of Theorem 1. Since analyticity is a local property, we may assume without loss of generality that V and W are open subsets of Euclidean spaces. Introducing dummy variables into the space of lower dimension, we may suppose that both V and W lie in E^n for some integer n, and that they both contain the origin O. We need only prove that f is analytic at (O, O).

We introduce the usual notation for partial derivatives, setting

$$D_j = i^{-1}\frac{\partial}{\partial x_j}$$

for $1 \leqslant j \leqslant n$,

$$D^\alpha = \prod_{j=1}^{n} D_j^{\alpha_j}$$

for any n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of non-negative integers,

$$\alpha = \sum_{j=1}^{n} \alpha_j, \alpha! = \prod_{j=1}^{n} (\alpha_j)!$$

If h is a function of two variables x and y in E^n, we indicate derivatives with respect to the x-variables by D_x^α and derivatives with respect to the y-variables by D_y^β.

If condition (A) holds, it follows immediately from the Cauchy integral formula for polycylinders that on some neighbourhood of (O, O) we have inequalities similar to those of condition (B), that is,

(C) *There exists a constant c_0 such that for all α and β,*

$$|D_x^\alpha f(x; y)| \leqslant c_0^{|\alpha|}\alpha! \leqslant C_0^{|\alpha|}|\alpha|!$$
$$|D_y^\beta f(x, y)| \leqslant c_0^{|\beta|}\beta! \leqslant C_0^{|\beta|}|\beta|!$$

Since f is Borel measurable on $V \times W$ and uniformly bounded on the neighbourhood N for which the bounds of (C) hold, it follows easily that its distribution derivatives $D_x^\alpha f$ and $D_y^\beta f$ coincide with the derivatives of f taken as a function of x with y held fixed and of y with x held fixed. It follows then from the bounds of condition (C) that on a fixed neighbourhood of

(O, O), all the derivatives $D_x^\alpha f$ and $D_y^\beta f$ are bounded functions. In particular, we know that $\{(\Delta_x)^r + (\Delta_y)^r\}f$ is bounded for every positive integer r, $(\Delta = -\sum_j D_j^2)$. It follows from the standard regularity theorems for solutions of elliptic partial differential equations that $D_x^\alpha D_y^\beta f$ lies in $L_{loc}^2(N)$ for every α and β, and hence by the Sobolev Imbedding Theorem (Schwartz **(4)**, vol. 2) f is infinitely differentiable in N.

We now remark that to prove that f is analytic at (O, O), it suffices to show the following:

(C′) *There exists a constant c_1 such that for all α and β,*

$$|(D_x^\alpha D_y^\beta f)(O, O)| \leqslant c_1^{|\alpha|+|\beta|}(|\alpha|+|\beta|)!$$

Indeed, suppose that (C′) holds. Then we may form the power series

$$f_1(x, y) = \sum_{\alpha,\beta} (\alpha!)^{-1}(\beta!)^{-1}(D_x^\alpha D_y^\beta f)(O, O)x^\alpha y^\beta,$$

where

$$x^\alpha = \prod_{j=1}^{n} x^{\alpha_j}, \; y^\beta = \prod_{j=1}^{n} y^{\beta_j}.$$

If the inequalities (C′) hold, the power series for f_1 may be majorized by the series

$$\sum_{\alpha,\beta} (\alpha!)^{-1}(\beta!)^{-1}c_1^{|\alpha|+|\beta|}(|\alpha|+|\beta|)!|x|^\alpha|y|^\beta,$$

where

$$|x|^\alpha = \prod_j |x_j|^{\alpha_j}.$$

The majorizing series is the expansion of the function

$$\left(1-\sum_j c_1|x|_j+c_1|y|_j\right)^{-1}$$

which converges for x and y sufficiently small. It follows that $f_1(x, y)$ is an analytic function of (x, y) on a suitably small neighbourhood of the origin (O, O). On the other hand, for every α and β,

$$(D_x^\alpha D_y^\beta f)(O, O) = (D_x^\alpha D_y^\beta f_1)(O, O).$$

The functions $f(\cdot, y)$ and $f_1(\cdot, y)$ are both analytic on a neighbourhood of O and have the same x-derivatives at O. Hence $f(x, O) = f_1(x, O)$ for $|x| \leqslant d_0$. Similarly, $D_y^\beta f(x, O) = (D_y^\beta f_1)(x, O)$ for $|x| \leqslant d_0$. Finally, for x held fixed, $f(x, \cdot)$ and $f_1(x, \cdot)$ are both analytic in a neighbourhood of the origin and have the same derivatives at $y = O$. Hence $f(x, y) = f_1(x, y)$. for all (x, y) near (O, O). Thus in order to show that f is analytic near (O, O) it clearly suffices to prove the inequalities (C′) for some value of the constant c_1.

To carry through the latter proof, we may assume after making a linear change of variables (which will only affect the constant c_0 in condition (C)) that the inequalities of (C) hold on the cube

$$R = \{|x_j| < 1, \quad |y_j| < 1, \quad 1 \leqslant j \leqslant n\}.$$

We define the function of a single real variable

$$\zeta_0(s) = \begin{cases} 0, & |s| \geqslant 1 \\ 1, & |s| < \frac{1}{2} \\ 2 - 2s, & \frac{1}{2} < |s| < 1. \end{cases}$$

The function ζ_0 is clearly continuous and piecewise continuously differentiable. Using ζ_0, we define the auxiliary function $\zeta(x, y)$ on $E^n \times E^n$ by

$$\zeta(x, y) = \prod_{j=1}^{n} \zeta_0(x_j) \cdot \prod_{j=1}^{n} \zeta_0(y_j).$$

For each positive integer r, we let

$$\zeta_r(x, y) = (\zeta(x, y))^{r+2}.$$

For each r, the function $\zeta_r(x, y)$ is $(r + 1)$-times continuously differentiable on $E^n \times E^n$ and has its support contained in R.

As another piece of auxiliary equipment, we consider for each sufficiently large integer m, the partial differential operator with constant coefficients on $E^n \times E^n$ defined by

$$A_{2m} = (-\Delta_x)^m + (-\Delta_y)^m + 1,$$

We construct an elementary solution for A_{2m} on $E^n \times E^n$ by a Fourier transform. Let

$$\langle x, \xi \rangle = \sum_j x_j \xi_j$$

for x and ξ in E^n. Then we define the function $e_{2m}(x, y)$ on $E^n \times E^n$ for $2m > 2n$ by

$$(1) \quad e_{2m}(x, y) = \int_{E^n} \int_{E^n} \exp(i\langle x, \xi \rangle + i\langle y, \xi' \rangle)(|\xi|^{2m} + |\xi'|^{2m} + 1)^{-1} d\xi \, d\xi'.$$

The integral converges uniformly on $E^n \times E^n$. We verify immediately by taking Fourier transforms (in the sense of tempered distributions of Schwartz **(4)**) that for every function v in $C^{2m}(E^n \times E^n)$ with support in R, we have

$$(2) \quad v(x, y) = \int_R e_{2m}(x - x_1, y - y_1)(A_{2m}v)(x_1, y_1) dx_1 dy_1; \ ((x, y) \in R)).$$

Let α and β be two indices of differentiation with $|\alpha| + |\beta| < 2m - 2n$. We verify by inspection that we can differentiate the integral defining e_{2m} in equation (1), $(|\alpha| + |\beta|)$-times obtaining

$$(3) \quad (D_x^\alpha D_y^\beta e_{2m})(x, y) = \int_{E^n} \int_{E^n} \xi^\alpha (\xi')^\beta \exp(i\langle x, \xi \rangle + i\langle y, \xi' \rangle)(|\xi|^{2m} + |\xi'|^{2m} + 1)^{-1} d\xi \, d\xi'.$$

It follows from (3) that $(D_x^\alpha D_y^\beta e_{2m})$ is continuous and bounded for $|\alpha| + |\beta| < 2m - 2n$, with

$$(4)\quad |D_x^\alpha D_y^\beta e_{2m}(x, y)| \leqslant \int_{E^n}\int_{E^n} |\xi|^{|\alpha|}|\xi'|^{|\beta|}(|\xi|^{2m} + |\xi'|^{2m} + 1)^{-1} d\xi d\xi' = K(m, n)$$

where, by the inequality

$$(5)\qquad |\xi|^{|\alpha|}|\xi'|^{|\beta|} \leqslant \frac{\alpha}{\alpha+\beta}|\xi|^{|\alpha|+|\beta|} + \frac{\beta}{\alpha+\beta}|\xi'|^{|\alpha|+|\beta|},$$

we know that

$$(6)\quad K(m, n) \leqslant \int_{E^n}\int_{E^n} (|\xi|^{2m-2n-1} + |\xi'|^{2m-2n-1} + 1)(|\xi|^{2m} + |\xi'|^{2m} + 1)^{-1}\, d\xi\, d\xi',$$

and the integral on the right side of the inequality (6) is bounded by

$$(7)\qquad 5\int_{E^n}\int_{E^n} (|\xi|^{2n+1} + |\xi'|^{2n+1} + 1)^{-1}\, d\xi\, d\xi',$$

which is independent of m. Thus we have for all (x, y) in $E^n \times E^n$,

$$(8)\qquad |D_x^\alpha D_y^\beta e_{2m}(x, y)| \leqslant K(n)$$

for $|\alpha| + |\beta| < 2m - 2n$.

Differentiating the equation (2) under the integral sign $|\alpha| + |\beta|$ times, we obtain for $|\alpha| + |\beta| < 2m - 2n$,

$$(9)\qquad D_x^\alpha D_y^\beta v(x, y) = \int_R (D_x^\alpha D_y^\beta e_{2m})(x - x_1, y - y_1) A_{2m} v(x_1, y_1)\, dx_1\, dy_1,$$

which yields, since the measure of R is exactly 1, the inequality

$$(10)\qquad |D_x^\alpha D_y^\beta v(x, y)| \leqslant K(n) \sup_{(x_1, y_1)\in R} |A_{2m} v(x_1, y_1)|$$

for (x, y) in R and $|\alpha| + |\beta| < 2m - 2n$.

We apply the inequality (10) to the given function $f(x, y)$ by setting

$$v(x, y) = f(x, y)\zeta_{2m}(x, y).$$

We put $(x, y) = (O, O)$ on the left-hand side of the inequality (10) and obtain, since ζ_{2m} is identically equal to 1 on a neighbourhood of (O, O), that

$$(11)\qquad |(D_x^\alpha D_y^\beta f)(O, O)| \leqslant K(n) \sup_{(x, y)\in R} |A_{2m}(\zeta_{2m}(x, y) \cdot f(x, y))|.$$

Let us examine the term on the right of the last inequality. It follows from the definition of A_{2m} that

$$A_{2m}(\zeta_{2m} f) = \{(-\Delta_x)^m + (-\Delta_y)^m + 1\}(\zeta_{2m} f) = \zeta_{2m} \{(-\Delta_x)^m + (-\Delta_y)^m + 1\} f + R_1,$$

where the remainder term R_1 is of the form

$$(12)\qquad R_1 = \sum_{\substack{\alpha,\beta \\ |\alpha|+|\beta|=2m,\ |\alpha|>0}} C_{\alpha\beta} D_x^\alpha(\zeta_{2m}) D_x^\beta f + \sum_{\substack{\alpha,\beta \\ |\alpha|+|\beta|=2m,\ |\alpha|>0}} C_{\alpha\beta} D_y^\alpha(\zeta_{2m}) D_y^\beta f.$$

The derivatives of ζ_{2m} are considered only on the set where ζ_{2m} is equal neither to 0 nor to 1. On this set ζ_{2m} is a polynomial of degree $(2m+2)$ in each variable. It follows from an easy application of a well-known theorem of Bernstein that for all (x, y) in R, $|\alpha| > 0$,

$$|(D_x^\alpha \zeta_{2m})(x, y)| \leqslant (2m+2)^{|\alpha|} 2^{|\alpha|}, \tag{13}$$

and

$$|(D_y^\alpha \zeta_{2m})(x, y)| \leqslant (2m+2)^{|\alpha|} 2^{|\alpha|}. \tag{14}$$

We now estimate R_1. The expansion of $(-\Delta_x)^m$ in elementary differential operators has n^m terms with coefficient 1. Each of these terms applied to a product $w_1 w_2$ can be written as the sum of 2^{2m} terms of the form $D^\alpha w_1 D^\beta w_2$ with coefficient 1. Hence the sum of the coefficeints

$$\sum_{\alpha,\beta} c_{\alpha,\beta}$$

in equation (12) is less than $2^{2m} n^m$. By the condition (C) and inequalities (13) and (14), it follows therefore that (assuming $c_0 > 1$)

$$|R_1| \leqslant 2(2n)^{2m} C_0^{2m} (2m+2)^{2m} 2^{2m}. \tag{15}$$

A similar, but sharper, estimate holds for the term $\zeta_{2m} A_{2m} f$. Combining these estimates, we obtain finally for $|\alpha| + |\beta| < 2m - 2n$

$$|(D_x^\alpha D_y^\beta f)(O, O)| \leqslant k_0 (4nC_0)^{2m} (2m)^{2m} \left(1 + \frac{2}{2m}\right)^{2m}. \tag{16}$$

By Stirling's formula, however, we know that

$$(2m)! \sim (4\pi m)^{\frac{1}{2}} (2m)^{2m} e^{-2m}.$$

Hence there exists an absolute constant k such that

$$(2m)^{2m} \leqslant k e^{2m} (2m)!$$

We obtain,

$$|(D_x^\alpha D_y^\beta f)(O, O)| \leqslant k\, k_0 e^2 (4nC_0 e)^{2m} (2m)! \leqslant (4nC_0 e)^{2m} \frac{(2m)!}{(2m-n-1)!} \tag{17}$$

$$(2m-n-1!) c_1^{2m-2n-1} (2m-n-1)!$$

if $|\alpha| + |\beta| = 2m - 2n - 1$, and similarly,

$$|(D_x^\alpha D_y^\beta f)(O, O)| \leqslant c_1^{2m-n-2} (2m-n-2)! \tag{18}$$

if $|\alpha| + |\beta| = 2m - 2n - 2$.

In particular, for any given α and β, we may choose m such that either $2m = |\alpha| + |\beta| + 2n + 1$, or $2m = |\alpha| + |\beta| + 2n + 2$. Since the inequalities (17) and (18) are therefore equivalent to the inequalities of condition (C′) for various α and β, the proof of Theorem 1 is therefore complete.

THEOREM 2. *Let f be an infinitely differentiable, separately analytic function on $V \times W$, where V and W are real analytic manifolds. Then there exists an everywhere dense open subset G of $V \times W$ such that f is analytic on G.*

Proof of Theorem 2. It clearly suffices to suppose that V and W are both cubes with centre at the origin in E^n and to show that f is analytic in some open subset of $V \times W = R$.

For each positive integer M, let

$$S_M = \{(x, y): (x, y) \in R, |D_x^\alpha f(x, y)| \leqslant M^{|\alpha|}\alpha! \quad \text{for all } \alpha\},$$

$$S_M' = \{(x, y): (x, y) \in R, |D_y^\alpha f(x, y)| \leqslant M^{|\alpha|}\alpha! \quad \text{for all } \alpha.\}$$

By the separate analyticity of the function f, each point (x, y) of R belongs to at least one of the sets S_M and at least one of the sets S_{M_1}'. By the continuity of the derivatives of f, each S_M is the intersection of closed sets and hence closed. Similarly each S_{M_1}' is closed, and so is

$$S_M \cap S_{M_1}'.$$

R is the union of the sets

$$S_M \cap S_{M_1}',$$

and by the Baire category theorem, one of these closed sets must have an interior. Let R' be a disk in

$$S_M \cap S_{M_1}'.$$

Then if M_2 is the larger of M and M_1, we have for all points of R' and all α,

$$|D_x^\alpha f(x, y)| \leqslant M_2^{|\alpha|}\alpha!$$

$$|D_y^\alpha f(x, y)| \leqslant M_2^{|\alpha|}\alpha!$$

It follows by the proof of Theorem 1 that f is then analytic at each interior point of R', and the proof of Theorem 2 is complete.

REFERENCES

1. J. de Barros-Neto, thesis, University of Sao Paulo (1960).
2. J. de Barros-Neto and F. E. Browder, *The analyticity of kernels*, Can. J. Math., *13* (1961), 645–649.
3. S. Bochner and W. T. Martin, *Several complex variables* (Princeton, 1949).
4. L. Schwartz, *Théorie des distributions* (Paris, 1950–1), I and II.

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UNIFORM CONTINUITY OF CONTINUOUS FUNCTIONS ON UNIFORM SPACES

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1. Introduction. Recently several topologists have called attention to the uniform structures (in most cases, the coarsest ones) under which every continuous real function is uniformly continuous (let us call the structures the [coarsest] *uc-structures*), and some important results have been found which closely relate, explicitly or implicitly, to the uc-structures, such as in the υS of Hewitt **(3)** and in the e-complete space of Shirota **(7)**. Under these circumstances it will be natural to pose, as Hitotumatu did **(4)**, the problem: which are the uniform spaces with the uc-structures? In **(1; 2)**, we characterized the metric spaces with such structures, and in this paper we shall give a solution to the problem in uniform spaces (§ 1), together with some of its applications to normal uniform spaces and to the products of metric spaces (§ 2). It is evident that every continuous real function on a uniform space is uniformly continuous if and only if the uniform structure of the space is finer than the uniform structure defined by all continuous real functions on the space. Our main theorem (Theorem 1) shows an internal aspect of this necessary and sufficient condition, and thus it permits us to construct the coarsest uc-structures internally (Corollary to Theorem 1). Then it is shown that the coarsest uc-structures are equivalent to the e-structures of Shirota **(7)** on some kind of spaces (Theorem 2). The last section of this paper is devoted to characterizing, as it were, uniformly pseudo-compact spaces, a generalization of pseudo-compact spaces in which Theorem 1 is reduced to a known result, and the characterization is a generalization of Theorem 2 of **(2)**.

2. Main theorem. We shall first give some definitions used in this section. A *function* is a real-valued continuous mapping, a *space* S is, unless otherwise specified, a uniform space, and a space is said to be *uc* if every function on the space is uniformly continuous. A sequence of subsets is *discrete* if any point of the space has a neighbourhood intersecting at most one member of the sequence, and it is *uniformly discrete* if there is an entourage V of the uniform structure such that $V(x)$ meets at most one member of it for any point x of the space. A sequence of subsets $\{A_n\}$ is *discretely normally separated* by a sequence of subsets $\{B_n\}$ if $\{B_n\}$ is discrete and there is a function f, $0 \leqslant f \leqslant 1$, for each n with values 1 on A_n and 0 on the complement of B_n, let us call the function the *characteristic function for* n. $\{A_n\}$ is *uniformly separated* by $\{B_n\}$ if there is an entourage V with $V(A_n) \subset B_n$ for all n.

THEOREM 1. *A uniform space is uc if and only if any sequence of subsets*

$\{A_n\}$, discretely normally separated by some sequence of subsets $\{B_n\}$, is uniformly separated by $\{B_n\}$.

Proof. Suppose the space is uc and f_n a characteristic function for n, then $f = \sup_n \inf(n, 2nf_n)$ and $g = \sup_n f_n$ are continuous, and so uniformly continuous, that is, there is an entourage V such that $(x, y) \in V$ implies $|f(x) - f(y)| < \frac{1}{2}$ and $|g(x) - g(y)| < \frac{1}{2}$. If x is in A_n, then $n - \frac{1}{2} < f(y) < n + \frac{1}{2}$ and $\frac{1}{2} < g(y)$. $g(y) \neq 0$ follows $y \in B_m$ for some m, and thus we have $g(y) = f_m(y)$, $\frac{1}{2} < f_m(y)$, $m < 2mf_m(y)$, $f(y) = m$, $n - \frac{1}{2} < m < n + \frac{1}{2}$, $m = n$, which means $V(A_n) \subset B_n$. For the converse, suppose f is a function on the space, consider the following closed intervals in the real line (n and k are integers and positive integers respectively):

$$A_n' = [(8n + 1)/6k, \quad (8n + 5)/6k],$$

$$B_n' = [4n/3k, (4n + 3)/3k],$$

and put $A_n = f^{-1}(A_n')$, $B_n = f^{-1}(B_n')$. Since $\{A_n\}$ is discretely normally separated by $\{B_n\}$, there is an entourage V_1 with $V_1(A_n) \subset B_n$ for all n. Similarly, for the closed intervals

$$C_n' = [(8n + 5)/6k, (8n + 9)/6k],$$

$$D_n' = [(4n + 2)/3k, (4n + 5)/3k],$$

and the sets $C_n = f^{-1}(C_n')$, $D_n = f^{-1}(D_n')$, there is an entourage V_2 with $V_2(C_n) \subset D_n$. Any point of the space belongs to some $A_n \cup C_n$. Let $(x, y) \in V \subset V_1 \cap V_2$, and let x belong to A_n, then y is in B_n and $|f(x) - f(y)| \leqslant 1/k$; similarly for C_n.

COROLLARY. *For any topological space S, the coarsest uc-structure $\mathscr{U}$ is generated by the relations*

$$U = (S - \cup A_n) \times (S - \cup A_n) \cup \cup_n (B_n \times B_n),$$

where $\{A_n\}$ is any sequence of closed subsets, discretely normally separated by the sequence $\{B_n\}$ of open subsets.

*Proof.** If U is as stated and $V \in \mathscr{U}$ taken such that $V(A_n) \subset B_n$ for all n (by Theorem 1), then $V \subset U$; in fact, if $(x, y) \in V$ and x or y lies in $\cup A_n$, then $(x, y) \in B_k \times B_k$ for some k, and otherwise $(x, y) \in (S - \cup A_n) \times (S - \cup A_n)$, this shows $V \subset U$, and hence $U \in \mathscr{U}$. On the other hand, if $W = \{(x, y); |f(x) - f(y)| \leqslant e\}$ for some function f on S and some real $e > 0$, then there are, by the proof of Theorem 1, two relations U and U' of the type considered such that $U \cap U' \subset W$. Since these W generate $\mathscr{U}$, the same holds for the U.

It is easily seen that the structure defined in this corollary is compatible with the original topology when the space is completely regular.

*This proof, neater than the original, is due to the referee.

Let us now give some spaces on which the coarsest uc-structure and the e-structure **(7)** are equivalent.

DEFINITION 1. *A topological space is* normally disconnected *if, for any open set G containing any closed set A, there is an open and closed set contained in G and containing A.*

THEOREM 2. *A coarsest uc-structure is equivalent to an e-structure in a T_1-space S if the space is normally disconnected and at most one-dimensional.*

Proof. Obviously a normally disconnected T_1-space is normal. Suppose that $\{X_n\}$ is a countable normal open covering of a normally disconnected T_1-space with at most one dimension, then we may assume $\{X_n\}$ has at most order 2. There is, for every n, an open and closed set H_n contained in X_n and containing $X_n - \bigcup_{i \neq n} X_i$. $K_1 = H_1$, $K_n = H_n - \bigcup_{i<n} H_i$ are open and closed sets which are disjoint from each other. $L_{ij} = X_i \cap X_j - (K_i \cup K_j)$ is open and closed, because, since $\{X_n\}$ is of at most order 2 and a point included only in X_i belongs to K_i, $L_{ij} = S - \bigcup_{h \neq i,j} X_h - (K_i \cup K_j)$. Consequently, $\{K_i, L_{ij}; i, j$ natural numbers$\}$ is an open and closed covering which refines $\{X_n\}$ and is discretely normally separated by itself.

3. Case of normal uniform spaces and others. Using the result of Dowker **(5**, Lemma 3**)**, we get the following statement which is a slight modification of Theorem 1: *In order that a normal space is uc it is necessary and sufficient that if $\{B_n\}$ is a disjoint sequence of open subsets, if closed $A_n \subset B_n$ for every n, and if $\bigcup A_n$ is closed, then there is an entourage V such that $V(A_n) \subset B_n$ for all n.* Furthermore we have

THEOREM 3. *A normal uniform space is uc if and only if any discrete sequence of subsets is uniformly discrete.*

Proof. Let $\{A_n\}$ be discrete in a normal uc-space, then we have, by a simple induction, a disjoint sequence of open subsets B_n containing $\bar{A}_n$ for every n, and, by the above remark, an entourage V such that $V(A_n) \subset B_n$, that is, $V(A_m) \cap A_n = \emptyset$. To prove the converse, let $\{B_n\}$ discretely normally separate $\{A_n\}$ in a space S satisfying the property in the assertion, then there are entourages V_1 and V_2 with $V_1(B_m) \cap B_n = \emptyset$, $m \neq n$, $V_2(\bigcup A_n) \cap (S - \bigcup B_n) = \emptyset$, and we have $V(A_n) \subset B_n$ for $V \subset V_1 \cap V_2$ and for each n.

DEFINITION 2. *A sequence of subsets $\{A_n\}$ is said to be* shrinking (*or* uniformly shrinking) *if A_n^0 (interior of A_n) $\supset \bar{A}_{n+1}$ (or $A_n \supset V(A_{n+1})$ for some entourage V) for every n.*

THEOREM 4. *A normal uniform space is uc if and only if any shrinking sequence of subsets $\{A_n\}$ with vacuous intersection is uniformly shrinking.*

Proof. Suppose a normal space S is uc and put $F_n = \bar{A}_{2n-1} - A_{2n}^0$, then

$\{F_n\}$ is discrete and there is a discrete sequence of open subsets $\{G_n\}$ **(5,** Lemma 3**)** with $F_n \subset G_n \subset A_{2n-2}{}^0 - \bar{A}_{2n+1}$, $n \geqslant 1$, $A_0 = S$. Similarly, for the discrete $\{H_n\}$, $H_n = \bar{A}_{2n} - A_{2n+1}{}^0$, there is a discrete $\{K_n\}$ of open subsets with $H_n \subset K_n \subset A_{2n-1}{}^0 - \bar{A}_{2n+2}$. There is, by Theorem 1, an entourage V with $V(F_n) \subset G_n$ and $V(H_n) \subset K_n$ for all n. If a point x of A_n belongs to A_m and not to A_{m+1}, and when $m = 2k-1$, then $x \in F_k$, $V(x) \subset G_k \subset A_{2k-2}{}^0 = A_{m-1}{}^0 \subset A_{n-1}$; when $m = 2k$, then $x \in H_k$, $V(x) \subset K_k \subset A_{2k-1}{}^0 = A_{m-1}{}^0 \subset A_{n-1}$. In any case, we have $V(A_n) \subset A_{n-1}$. Conversely, for a discrete sequence of closed subsets $\{A_n\}$, there is an open discrete $\{B_n\}$ with $B_n \supset A_n$. For every n we can make up a sequence of $2n-2$ open sets $C_n{}^i$ such that $\{B_n, C_n{}^1, C_n{}^2, \ldots, C_n{}^{2n-2}, A_n\}$ is shrinking, and put

$$F_{2n-1} = A_n \cup \Big(\bigcup_{i>n} C_i^{2n-1} \Big),$$

$$F_{2n} = \bigcup_{i>n} C_i^{2n}.$$

Since $\{B_n\}$ is discrete, $\{F_n\}$ is shrinking, so that there is an entourage V with $V(F_n) \subset F_{n-1}$, that is, $V(A_m) \cap A_n = \emptyset$.

COROLLARY. *A normal space is uc if and only if for any decreasing sequence*

$$F_1 \supset G_1 \supset \ldots \supset F_n \supset G_n \supset \ldots$$

of closed F_n and open G_n with vacuous intersection, there is an entourage V with $G_{n-1} \supset V(F_n)$ for $n \geqslant 2$.

We shall now apply Theorem 1 to find the condition under which the product space of metric spaces is uc. We already know the various equivalent conditions a metric space to be uc **(1; 2)**, which are of course easily verified also by using the above theorems. Let us recall some of them for later use; for this, it is convenient to define the following. A subset A of a space S is said to be *uniformly isolated in S* if there is an entourage V in S such that for every point x in A, $V(x)$ contains no point of S except for x.

THEOREM 5 **(2,** Theorem 1**)**. *A metric space S is uc if and only if it satisfies one of the following equivalent conditions.*

(i) *A set of all but finitely many members of a discrete sequence of points in S is uniformly isolated in S.*

(ii) *A set of all but finitely many points of the subset which has no accumulation point in S is uniformly isolated in S.*

LEMMA. *If a product space $S = \Pi_{\alpha \in Z} S_\alpha$ is uc, then so also is $S' = \Pi_{\alpha \in Y} S_\alpha$ for any $Y \subset Z$.*

Proof. We may suppose S' is a uniform subspace of S. We can make up, from a function f on S', an extension g over S by defining $g(x) = f(x')$ for every x of S, where x' is a projection of x on S'.

THEOREM 6. *A product space S of infinitely many metric spaces S_α is uc if and only if*

(i) *S is compact, or*

(ii) *all but finitely many factor spaces are one-point-spaces and* (1) *all are uniformly isolated or* (2) *all are finite except for one which is a non-compact and non-uniformly isolated uc-space.*

Proof. The verification of the "if" part is obvious, and is hence passed over. Let us suppose S is uc and not compact, and so there is a factor space, say S_0, including a discrete sequence $\{x_n\}$ of points; we may then, by the Lemma and Theorem 5, assume that every x_n is isolated. When there is a countable number of spaces $S_1, S_2, \ldots$, every one of which includes two points at least, we take a neighbourhood G_n of a point y_n in S_n for every $n \neq 0$ such that S_n includes a point which does not belong to G_n.

$$H_n = (x_n) \times G_1 \times \ldots \times G_n \times \prod_{a \in Z(n)} S_a,$$

where $Z(n)$ is the set of all indices excepting $0, 1, \ldots, n$, is a neighbourhood of a point p_n of S whose projection on S_0 (or S_i, $1 \leqslant i \leqslant n$) is x_n (or y_i). $\{p_n\}$ is discretely normally separated by $\{H_n\}$, but we cannot find an entourage V in S such that $V(p_n) \subset H_n$ for all n. Consequently all but finitely many factor spaces, for example, excepting $S_0, S_1, \ldots, S_n$, are one-point-spaces. If S_i, $i \neq 0$, has an accumulation point, then $S_0 \times S_i$ includes a discrete sequence of points which are not isolated, this, by Theorem 5, contradicts the Lemma. Therefore all S_i except S_0 have no accumulation point, that is, uniformly isolated by Theorem 5. If S_0 has no accumulation point, it is also uniformly isolated. If S_0 has an accumulation point, S_i, $i \neq 0$, cannot have infinitely many points just as above.

4. Remarks. The condition in order that every *bounded* function on a space is uniformly continuous is well known **(6)**, so Theorem 1 is essentially new for a non pseudo-compact space. In a uc-space, pseudo-compactness follows from precompactness, which is generalized in the following definition. The generalized concept closely relates to a property of a uniform space, which may be called uniform pseudo-compactness, as shown in Theorem 7, a generalization of Theorem 2 of **(2)**.

DEFINITION 3. *Let V be an entourage. The finite sequence of points $x_0, x_1, \ldots, x_m$ satisfying $(x_{i-1}, x_i) \in V$ is said to be a* V-chain with length m. *If for any entourage V there are finitely many points $p_1, \ldots, p_j$ and a positive integer m such that every point of the space can be bound with some p_i by a V-chain with length m, that is,*

$$\bigcup_{i=1}^{j} V^m(p_i) = S,$$

then the space is said to be finitely chainable.

THEOREM 7. *A uniform space is finitely chainable if and only if every uniformly continuous function is bounded.*

Proof. The verification is essentially similar to that of Theorem 2 of **(2)**. We shall verify the "if" part only. Suppose a uniform space is not finitely chainable, and so there is an entourage V such that for any finite number of points and any positive integer n there is a point which cannot be bound with any one of the points selected above by a V-chain with length n. Let A_0^0 be a set consisting of a fixed point, and

$$A_0^n = V(A_0^{n-1}),\ V_0 = V,\ V_nV_n \subset V_{n-1},\ V_n^{-1} = V_n,$$

$$U_t = V_{n_m}V_{n_m-1}\dots V_{n_1},\ t = \sum_{i=1}^{m} 2^{-n_i},\ n_1 < n_2 < \dots < n_m,$$

then the function

$$f_n(x) = \begin{cases} \sup\{t; x \notin U_t(A_0^{n-1})\}, \\ 0 \quad \text{on } A_0^{n-1}, \end{cases}$$

is uniformly continuous (cf. **(8**, proof of Theorem 1, p. 13**)**).

(1) When $A_0^n \neq A_0^{n-1}$ for all n. Let us put $f(x) = n - 1 + f_n(x)$ for x belonging to A_0^n and not to A_0^{n-1}, then $f(x)$ is uniformly continuous on $A_0 = \bigcup_n A_0^n$. To see this, let us suppose y is in $V_m(x)$. Then y belongs to A_0^{n+1} and not to A_0^{n-2}. (i) If y is in A_0^{n-1}, then $f(y) = n - 2 + f_{n-1}(y) \leqslant f(x)$; $f_{n-1}(y) < 1 - 2^{-m} = t$ implies $y \in U_t(A_0^{n-2})$, $x \in V_mU_t(A_0^{n-2}) \subset A_0^{n-1}$, a contradiction; so we have $f_{n-1}(y) \geqslant 1 - 2^{-m}$. Therefore $|f(x) - f(y)| = f(x) - f(y) \leqslant 2^{-m+1}$. (ii) If y belongs to A_0^n and not to A_0^{n-1}, then $f(y) = n - 1 + f_n(y)$, $|f(x) - f(y)| = |f_n(x) - f_n(y)| < 2^{-m+1}$. (iii) The remaining case for y is similar to (i).

(2) When $A_0^n = A_0^{n-1}$ for some n. If we can make up an unbounded function which is uniformly continuous on A_1 constructed from a point of the complement of A_0 in the similar way to (1), our proof will be complete.

(3) When we cannot get a desired function on A_m obtained in the same way with (2) for every natural number m, then we put $f(x) = m$ for x in A_m and $= 0$ otherwise. Since the space is not finitely chainable, the function is unbounded and uniformly continuous.

References

1. M. Atsuji, *Solution of problem 6.2.16 (in case of metric space)* (Japanese), Sugaku of Math. Soc. Japan, *8* (1957), 152–153.
2. ——— *Uniform continuity of continuous functions of metric spaces*, Pacific J. Math., *8* (1958), 11–16.
3. E. Hewitt, *Rings of real-valued continuous functions, I*, Trans. Amer. Math. Soc., *64* (1948), 45–99.
4. S. Hitotumatu, *Problem 6.2.16* (Japanese), Sugaku of Math. Soc. Japan, *6* (1954), 120.
5. E. Michael, *Another note on paracompact spaces*, Proc. Amer. Math. Soc., *8* (1957), 822–828.

6. J. Nagata, *On the uniform topology of bicompactifications*, J. Inst. Polytech. Osaka City Univ., *1* (1950), 28–38.
7. T. Shirota, *A class of topological spaces*, Osaka Math. J., *4* (1952), 23–40.
8. A. Weil, *Sur les espaces à structure uniform et sur la topologie générale* (Paris: Hermann, 1937).

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ON NORMED ALGEBRAS WHOSE NORMS SATISFY POLYNOMIAL IDENTITIES

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1. Introduction. In this note we are concerned with normed algebras over a non-discrete field with absolute value. The norm, N, of a normed algebra A is said to *satisfy a polynomial identity* on a subset B if there is a polynomial $P(\zeta_1, \ldots, \zeta_r)$ such that $P(N(x_1), \ldots, N(x_r)) = N(P(x_1, \ldots, x_r))$ whenever $x_1, \ldots, x_r$ are in B, where the polynomial has rational integer coefficients, degree greater than 1, constant term zero, and non-negative coefficients for each term of highest degree. It is shown in Theorem 1, following a method of proof used by Kadison in **(6, § 7)**, that if the norm of a normed algebra satisfies a polynomial identity on the entire algebra, then the norm is power multiplicative. (That is, then $N(x)^2 = N(x^2)$ for all x.)

As incidental by-products of this result, some corollaries are obtained which are related to Ostrowski's results in **(8)** and to Mazur's Theorem **(7**, Theorem 1**)**. For example, Corollary 3 of Theorem 1 shows that if a normed division algebra A, over the real field normed with some power of its ordinary absolute value, has a norm which is stable (in the sense of **(3)**) and which satisfies a polynomial identity on A, then A is isomorphic, as a real algebra, with the real field, or the complex field, or the division ring of all real quaternions. (Compare with Mazur's Theorem 1 in **(7)**.)

Gelfand's algebraic characterization **(5)** of a semi-simple, commutative, complex Banach algebra with unit element, as an algebra of some of the continuous complex-valued functions on a suitable compact space, is well known. (*Compact* spaces throughout this note will be Hausdorff spaces for which every open covering has a finite subcovering; that is, we follow the definition employed in **(4)**.) A related result is obtained in Theorem 2 of this note, where it is shown that if a connected, commutative normed algebra A over a non-discrete field contains a non-zero element j such that $j^2x + x = 0$ for all x, and if the norm of A satisfies a polynomial identity on A, then A is algebraically and topologically isomorphic to a ring of some of the continuous complex-valued functions on a suitable compact space. Here the assumption that the norm satisfies a polynomial identity is stronger than assuming semi-simplicity of the algebra, while the assumption that the element j exists is weaker than assuming that the algebra is a complex algebra with unit e, since $i \cdot e$ could be taken as j in a complex algebra with unit.

One particular consequence of Gelfand's result quoted above was that **(5,**

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Satz 16, Folgerung 3) a commutative, complex Banach algebra A with unit element, such that the norm of A satisfies the condition $N(x)^2 = N(x^2)$ for all x in A, may be identified with the algebra of all continuous complex-valued functions (with the usual supremum norm) on a suitable compact space, provided that A "contains conjugate functions" of all its elements. In Theorem 3 of this note a similar result is obtained under much weaker assumptions; it is shown that a complete normed algebra which satisfies the hypotheses of Theorem 2 (given in the preceding paragraph), and which "contains conjugate functions" in an appropriate sense, must be algebraically and topologically isomorphic to the ring of all continuous complex-valued functions on a suitable compact space.

An interesting feature of the proofs is the use of Ostrowski's results **(8)** at several points where the standard practice in obtaining comparable results for Banach algebras is to rely on Mazur's Theorem **(7)**. This change appears necessary since our hypotheses do not seem to permit the immediate use of Mazur's Theorem.

2. Preliminaries. We shall consider in the sequel normed algebras, usually with unit element e, over non-discrete fields with absolute value. (For basic terminology, see **(4**, chapter IX, § 3, no. 7**)**.) The normed algebras to be studied will be subjected to conditions on their norms similar to those imposed in **(3)**; it is assumed that the material contained in that paper is known to the reader.

Every normed algebra is a metric ring; such words as *homomorphism* and *isomorphism* will be understood to refer only to the ring structure of the algebras under consideration, unless the contrary is indicated. A norm-preserving isomorphism of a metric ring R into a metric ring R' will be called an *isometry* of R into R'; if there is an isometry of a metric ring R onto a metric ring R' we shall say that R is *isometric* to R'.

A polynomial $P(\zeta_1, \ldots, \zeta_r)$ in the r indeterminates $\zeta_1, \ldots, \zeta_r$ will be called *regular* if it has rational integer coefficients, constant term zero, degree greater than 1, and if every term of highest degree has a non-negative coefficient. The number of indeterminates which actually appear in a regular polynomial will be called the *rank* of the polynomial, while the degree of the terms of lowest degree in a regular polynomial will be called the *order* of the polynomial. It may be noted that the rank and the order of a regular polynomial are both positive integers.

In every ring it is possible to define, in an obvious way, multiplication of ring elements by rational integers. It follows easily that if $P(\zeta_1, \ldots, \zeta_r)$ is a regular polynomial, and if $x_1, \ldots, x_r$ are elements of a ring R, then there is defined an element $P(x_1, \ldots, x_r)$ belonging to R. If N is a pseudonorm for a ring R and B is a subset of R such that there is a regular polynomial $P(\zeta_1, \ldots, \zeta_r)$ for which $P(N(x_1), \ldots, N(x_r)) = N(P(x_1, \ldots, x_r))$ whenever $x_1, \ldots, x_r$ are in B, then we shall say that N *satisfies a polynomial identity*

on B. Any terminology applicable to regular polynomials will also be applied to situations in which a pseudonorm satisfies a polynomial identity, in order to describe the polynomial involved in the identity.

Polynomial identities have been imposed upon the norm of a metric ring in several instances in the past. For example, Ostrowski confined himself in **(8)** to fields having a norm N such that $N(x_1)N(x_2) = N(x_1x_2)$ for all x_1 and x_2; this is equivalent to assuming that N satisfies on the whole field the polynomial identity involving the regular polynomial $P(\zeta_1, \zeta_2) \equiv \zeta_1 \cdot \zeta_2$. In **(3)** pseudonorms N were considered such that $N(x)^2 = N(x^2)$ for all x, and this is the same as saying that N satisfies on the whole ring the polynomial identity corresponding to the polynomial $P(\zeta) \equiv \zeta^2$. This latter polynomial identity is quite typical, and we shall show that, under reasonably general conditions, the norm of a normed algebra must satisfy this particular polynomial identity on the algebra if it satisfies a polynomial identity on the algebra.

First, we note that if a pseudonorm N satisfies a polynomial identity on a set B, then N satisfies a polynomial identity of rank one on B. For, if $P(N(x_1), \ldots, N(x_r)) = N(P(x_1, \ldots, x_r))$ whenever $x_1, \ldots, x_r$ are in B, then $P(N(x), \ldots, N(x)) = N(P(x, \ldots, x))$ for all x in B; that is, N satisfies the polynomial identity $Q(N(x)) = N(Q(x))$ on B, where $Q(\zeta) \equiv P(\zeta, \ldots, \zeta)$ is clearly a regular polynomial having the same degree as P, and with order at least as large as that of P. Thus, we may always pass from a general polynomial identity to one of rank 1. The next step is to show that, under appropriate conditions, we may then pass to the specific polynomial identity $N(x)^2 = N(x^2)$, which has already been investigated in **(3)**.

THEOREM 1. *Let A be a normed algebra, with norm N, over a non-discrete field K with absolute value, such that N satisfies a polynomial identity on A (or a polynomial identity of order greater than 1 on a neighbourhood of zero in A). Then N is power multiplicative; that is, $N(x)^2 = N(x^2)$ for all x in A.*

Proof. Because of the remarks made above, we may assume that the polynomial identity satisfied by N has rank one, say $P(N(x)) = N(P(x))$ for all x. If P has degree m and order n, with $P(\zeta) \equiv \alpha_0 \cdot \zeta^m + \ldots + \alpha_s \cdot \zeta^n$, where $s = m - n$, then for x in A and non-zero k in K, kx is in A, and therefore $P(N(kx)) = N(P(kx))$. If $|| \ ||$ is the absolute value for K, then $N(kx) = ||k|| \cdot N(x)$, so that $P(||k|| \cdot N(x)) = P(N(kx)) = N(P(kx))$ for x in A and k a non-zero element of K. If $P(||k|| \cdot N(x)) = N(P(kx))$ is written explicitly, we obtain:

$$(1) \quad \alpha_0 \cdot ||k||^m \cdot N(x)^m + \ldots + \alpha_s \cdot ||k||^n \cdot N(x)^n = N(\alpha_0 \cdot k^m \cdot x^m + \ldots + \alpha_s \cdot k^n \cdot x^n).$$

If we divide by $||k||^m = ||k^m||$, we have:

$$(2) \quad \alpha_0 \cdot N(x)^m + \ldots + \alpha_s \cdot ||k||^{-s} \cdot N(x)^n = N(\alpha_0 \cdot x^m + \ldots + \alpha_s \cdot k^{-s} \cdot x^n),$$

where division by $||k^m||$ was effected on the right side of the equation by moving k^m within the parentheses and there carrying out the division by k^m.

Let $||k||$ tend to infinity in (2), so that $\alpha_0 \cdot N(x)^m = N(\alpha_0 \cdot x^m)$ for every fixed x in A. Then $\alpha_0 \cdot N(x)^m = N(\alpha_0 \cdot x^m) \leqslant \alpha_0 \cdot N(x^m) \leqslant \alpha_0 \cdot N(x)^{m-2} \cdot N(x^2)$, and division by $\alpha_0 \cdot N(x)^{m-2}$ yields $N(x)^2 \leqslant N(x^2)$. But $N(x^2) \leqslant N(x)^2$ always, whence $N(x)^2 = N(x^2)$ for all x; thus, N is power multiplicative.

In case we make the alternative assumption that N satisfies a polynomial identity of order greater than 1 on a neighbourhood U of zero in A, then for a fixed x in A we have kx in U whenever $||k||$ is sufficiently small. This means that (1) applies in this situation, with $n > 1$. The procedure is now similar to that followed previously, except that we work at the other end of the polynomial. Thus, we divide (1) by $||k||^n = ||k^n||$ and then let $||k||$ tend to zero. This gives us $\alpha_s \cdot N(x)^s = N(\alpha_s \cdot x^s)$ for all x.

N is obviously power multiplicative if A consists solely of the zero element, and it will therefore suffice to complete the proof only for the case in which A contains an element different from zero. If there is a non-zero x in A, then $\alpha_s \cdot N(x)^s \neq 0$, and therefore $N(\alpha_s \cdot x^s) \neq 0$ for such an x. Thus, the right side of the equation $\alpha_s \cdot N(x)^s = N(\alpha_s \cdot x^s)$ is positive, so that the left side is positive, and α_s is then positive. Then for every x in A we have $\alpha_s \cdot N(x)^s = N(\alpha_s \cdot x^s) \leqslant \alpha_s \cdot N(x^s) \leqslant \alpha_s \cdot N(x)^{s-2} N(x^2)$. Division by $\alpha_s \cdot N(x)^{s-2}$ shows that $N(x)^2 \leqslant N(x^2)$, and consequently $N(x)^2 = N(x^2)$ for all x. That is, N is power multiplicative.

COROLLARY 1. *Let A be an archimedean normed division algebra, over a non-discrete field K with absolute value. If the norm of A is stable and satisfies a polynomial identity on A (or a polynomial identity of order greater than 1 on a neighbourhood of zero in A), then A is isomorphic to a division subring of the division ring $\mathfrak{Q}$ of all real quaternions. If, in addition, A is commutative, then A is isomorphic to a subfield of the field $\mathbb{C}$ of all complex numbers.*

COROLLARY 2. *Let A be a connected normed division algebra, over a non-discrete field K with absolute value. If the norm for A is stable and satisfies a polynomial identity on A (or a polynomial identity of order greater than 1 on a neighbourhood of zero in A), then A is isomorphic to a division subring of $\mathfrak{Q}$. If, in addition, A is commutative, then A is isomorphic to a subfield of $\mathbb{C}$.*

The theorem shows that the norm is power multiplicative in both corollaries, so that the conclusions in the second sentences of these corollaries follow from (3, Theorem 8, and Corollary 2 of Theorem 6). The final sentence in each corollary follows from the fact that every commutative subring of $\mathfrak{Q}$ is contained in a maximal commutative subring, for it is easily demonstrated that every maximal commutative subring of $\mathfrak{Q}$ is isomorphic to $\mathbb{C}$.

An interesting special case of Corollary 2 occurs when the algebra is a normed algebra over the real field. First, if ρ is a real number such that $0 < \sigma \leqslant 1$, the symbol $\mathfrak{R}^{(\rho)}$ will denote the field of all real numbers, with

the ρth power of the ordinary absolute value taken as a norm for $\mathfrak{R}^{(\rho)}$. It is easily established that each $\mathfrak{R}^{(\rho)}$ is a connected, complete field with absolute value. Then every normed algebra over $\mathfrak{R}^{(\rho)}$ is connected since $\mathfrak{R}^{(\rho)}$ is connected. This would yield a special case of Corollary 2, but it is possible to obtain a stronger conclusion when connectedness of the algebra is replaced as a hypothesis by the assumption that the field K of scalars is a field $\mathfrak{R}^{(\rho)}$.

COROLLARY 3. *Let ρ be a real number such that $0 < \rho \leqslant 1$, and let A be a normed division algebra over $\mathfrak{R}^{(\rho)}$, such that the norm for A is stable and satisfies a polynomial identity on A (or a polynomial identity of order greater than 1 on a neighbourhood of zero in A). Then A is isomorphic, as a real algebra, to the algebra $\mathfrak{R}$ of all real numbers, or to $\mathfrak{C}$, or to $\mathfrak{Q}$.*

Proof. If N is the norm for A then N is power multiplicative, by Theorem 1, so that Corollary 2 of Theorem 2 in **(3)** shows that there is an absolute value N' subordinate to N.

Let α be a non-zero real number, and let e be the unit element of A. Then $N'(\alpha \cdot e) \leqslant N(\alpha \cdot e) = |\alpha|^{\rho} \cdot N(e) = |\alpha|^{\rho}$. Similarly, $N'(\alpha^{-1} \cdot e) \leqslant |\alpha^{-1}|^{\rho} = |\alpha|^{-\rho}$. But $1 = N'(e) = N'((\alpha \cdot e)(\alpha^{-1} \cdot e)) = N'(\alpha \cdot e)N'(\alpha^{-1} \cdot e) \leqslant |\alpha|^{\rho} \cdot |\alpha|^{-\rho} = 1$, so that $N'(\alpha \cdot e) = |\alpha|^{\rho}$. Thus, $N'(\alpha \cdot x) = N'((\alpha \cdot e)x) = N'(\alpha \cdot e)N'(x) = |\alpha|^{\rho} \cdot N(x)$ for every x in A. It follows easily that $N'(\alpha \cdot x) = |\alpha|^{\rho} \cdot N'(x)$ for every x in A and for every real scalar α. If A' is the structure obtained from A by using N' instead of N as the norm, then we have just shown that A' is a normed algebra over $\mathfrak{R}^{(\rho)}$.

Let A'' be the completion of A', so that A'' is also a normed algebra over $\mathfrak{R}^{(\rho)}$. Since A' is a division ring with absolute value, A'' is a division ring with absolute value. In addition, A'' is connected since it is a normed algebra over $\mathfrak{R}^{(\rho)}$; furthermore, A'' is complete. Theorem 11 of **(2)** then shows that there exists an algebraic and topological isomorphism σ of A'' onto $\mathfrak{R}$, or $\mathfrak{C}$, or $\mathfrak{Q}$. The continuity of σ implies that σ is not only a ring-isomorphism, but also an isomorphism as algebras over $\mathfrak{R}$. Since A'' is therefore a finite-dimensional algebra over $\mathfrak{R}$, it follows that the subalgebra A' is also a finite-dimensional algebra over $\mathfrak{R}$. The theorem of Frobenius then implies that the division algebra A' is isomorphic, as an algebra over $\mathfrak{R}$, to one of the algebras $\mathfrak{R}$, $\mathfrak{C}$, and $\mathfrak{Q}$. But A has the same underlying algebra as A', so that A is also isomorphic, as an algebra over $\mathfrak{R}$, to one of the algebras $\mathfrak{R}$, $\mathfrak{C}$, and $\mathfrak{Q}$.

Corollary 3 bears some similarity to Mazur's Theorem **(7**, Theorem 1**)**, but the latter also assumes that $\rho = 1$, while the hypotheses of Corollary 3 are stronger than those of Mazur's Theorem in assuming that the norm is stable and satisfies a polynomial identity. The standard proofs of Mazur's Theorem do not seem to be adaptable for proving the corollary, however, despite its similarity to Mazur's Theorem. Gelfand's proof in **(5)**, using analytic function theory, and Tornheim's elementary proof in **(10)** both require that the field of scalars be the real field or the complex field, equipped

with the ordinary absolute value. The method of proof employed by Arens in **(1)** also seems to require stronger conditions than are available in Corollary 3.

3. Representations as rings of continuous functions. Gelfand showed in **(5)** that a semi-simple, commutative, complex Banach algebra with unit may be identified algebraically with an algebra of some of the continuous complex-valued functions defined on a suitable compact space. We shall obtain a result which is somewhat similar, although our hypotheses will be weaker in most particulars, except for replacing the assumption of semi-simplicity by the stronger assumption that the norm of the algebra satisfies a polynomial identity on the algebra. (It is easily seen that if the norm of a normed algebra over a non-discrete field satisfies a polynomial identity on the algebra, then Theorem 1 shows that the norm is power multiplicative, so that the algebra contains no non-zero nilpotents and is therefore semi-simple.) Before proceeding to our representation theorem, we first introduce some pertinent definitions.

A central, non-zero element j of a ring R will be called a *gaussian element* of R if $j^2x + x = 0$ for all x in R. The existence of a gaussian element in a ring always implies that the ring has a unit element; for, if j is a gaussian element in a ring R, then $-j^2$ is a unit for R. In a complex algebra with unit e, the elements $i \cdot e$ and $(-i) \cdot e$ are examples of gaussian elements.

If ρ is a real number with $0 < \rho \leqslant 1$, the symbol $\mathfrak{C}^{(\rho)}$ will denote the field of all complex numbers, with the ρth power of the ordinary absolute value taken as the norm. Then $\mathfrak{C}^{(\rho)}$ is a connected, complete field with absolute value; in particular, $\mathfrak{C}^{(\rho)}$ is *archimedean*, in the sense of **(8)** or **(2)**. If K is any archimedean field with absolute value, then there is a real number ρ, with $0 < \rho \leqslant 1$, such that K is isometric to a subfield of $\mathfrak{C}^{(\rho)}$. (This result occurs essentially in **(8)**; an easy proof is obtained by completing K and then applying Theorem 11 of **(2)**.) The isometry of K into a $\mathfrak{C}^{(\rho)}$ is not necessarily unique, but its restriction to the prime field of K is unique, and ρ is consequently determined uniquely by K. In fact, if n is any integer greater than 1, then ρ is determined by the condition $\rho = \log_n(||n||)$ if the absolute value in K is denoted by $||\ ||$. We shall refer to the number ρ as the *exponent* for the archimedean field K. It is clear that every subfield of an archimedean field K has the same exponent as K. (Lemma 14 of **(2)** clearly implies that a subfield of an archimedean field with absolute value must also be archimedean; for, whether a field with absolute value is archimedean or not is determined by the behaviour of the absolute value on the prime field.)

If ρ is a real number with $0 < \rho \leqslant 1$, and if Φ is a compact space, the symbol $C(\Phi; \mathfrak{C}; \rho)$ will designate the set of all continuous complex-valued functions $x(\phi)$ defined on Φ, with algebraic operations defined in the obvious way, and with the norm N such that $N(x) = \sup\{|x(\phi)|^\rho \mid \phi \in \Phi\}$ for each x in $C(\Phi; \mathfrak{C}; \rho)$. Clearly, the norm in $C(\Phi; \mathfrak{C}; \rho)$ is simply the ρth power of

the usual supremum norm and is, therefore, power multiplicative. It is easily verified that $\mathfrak{C}(\Phi; \mathbb{C}; \rho)$ is a connected, commutative, complete normed algebra with unit, over the field $C^{(\rho)}$. The topology in $C(\Phi; \mathbb{C}; \rho)$ is evidently the topology of uniform convergence on Φ, and each $C(\Phi; \mathbb{C}; \rho)$ clearly contains gaussian elements.

It is now possible to prove that a connected, commutative normed algebra which contains gaussian elements, and which has a norm that satisfies a polynomial identity on the algebra, is isometric to a subring of a $C(\Phi; \mathbb{C}; \rho)$. This result is quite similar to the result of Gelfand which was described at the beginning of this section, but our hypotheses are clearly weaker than those of Gelfand, with the one exception already noted.

LEMMA 1. *Let A be a connected, commutative normed algebra, with norm N, over a non-discrete field K with absolute value. Suppose that N satisfies a polynomial identity on A (or a polynomial identity of order greater than 1 on a neighbourhood of zero in A). Then K is archimedean, and, if ρ is the exponent for K, it is possible to find for each non-zero c in A a non-zero homomorphism ϕ of A into $\mathfrak{C}^{(\rho)}$, such that $||\phi(x)|| \leqslant N(x)$ for all x in A, and $||\phi(c)|| = N(c)$.*

Proof. Theorem 1 shows that N is power multiplicative.

If c is a given non-zero element of A, let $\mathcal{N}$ be the set of all power multiplicative pseudonorms N' subordinate to N and such that: (i) $N'(c) = N(c)$, (ii) $N'(cx) = N'(c) \cdot N'(x)$ for all x in A, and (iii) $N'(kx) = ||k|| \cdot N'(x)$ whenever k is in K and x is in A. It is easily shown that N_c belongs to $\mathcal{N}$ and that $\mathcal{N}$ is a hereditary system. The method of proof of Lemma 4 in **(3)** may be used to show that $\mathcal{N}$ contains a minimal element N', and **(3,** Lemma 2**)** shows that N' is a pseudo absolute value. Then $\bar{A} = A/I(N')$ is an algebra over K; if $\bar{N}$ is the function on $\bar{A}$ such that the value $\bar{N}(X)$ assumed by $\bar{N}$ on the residue class X (modulo $I(N')$) is equal to the constant value assumed by N' on the elements of X, then $\bar{A}$ becomes a commutative normed algebra, with norm $\bar{N}$, over K. Also, $\bar{N}$ is an absolute value for $\bar{A}$, so that $\bar{A}$ has no proper zero-divisors. The natural mapping η of A onto $\bar{A}$ is a homomorphism of A onto $\bar{A}$, as algebras over K, and we have $\bar{N}(\eta(x)) = N'(x)$ for all x in A. Thus, $\bar{N}(\eta(c)) = N'(c) = N(c) \neq 0$, so that $\eta(c)$ is a non-zero element of $\bar{A}$.

Since $\bar{A}$ is a commutative non-zero algebra without proper zero-divisors, a standard procedure permits us to embed $\bar{A}$ in a field E of quotients. With the obvious norm $\bar{N}$ chosen for E, we obtain a normed algebra E over K, such that E is a field with absolute value. The natural mapping ξ of $\bar{A}$ into E is a norm-preserving isomorphism of $\bar{A}$ into E, as algebras over K. The inequality $\bar{N}(\eta(x) - \eta(y)) = \bar{N}(\eta(x - y)) = N'(x - y) \leqslant N(x - y)$ shows that η is continuous, so that $\bar{A}$ is connected since it is the continuous image of the connected set A. Also, $\bar{A}$ contains the distinct points 0 and $\eta(c)$, and the image of $\bar{A}$ under the continuous one-to-one mapping ξ is therefore a connected subset of E containing more than one point. Then E is a field with

absolute value and is not totally disconnected; Lemma 15 of **(2)** implies that E must then be archimedean. But K may be identified with a subfield of E in an obvious way, and it follows that K is archimedean.

Let ρ be the exponent for K, so that ρ is also the exponent for E. Then there exists an isometry θ of E into $\mathfrak{C}^{(\rho)}$, and we let ϕ be the mapping obtained by applying η, then ξ, and finally θ. Clearly, ϕ is a homomorphism of A into $\mathfrak{C}^{(\rho)}$, and $||\phi(x)|| = ||\theta(\xi(\eta(x)))|| = \bar{N}(\xi(\eta(x))) = \bar{N}(\eta(x)) = N'(x) \leqslant N(x)$ for all x in A. Also, $||\phi(c)|| = ||\theta(\xi(\eta(c)))|| = \bar{N}(\xi(\eta(c))) = \bar{N}(\eta(c)) = N'(c) = N(c) \neq 0$, and this shows that ϕ is not zero since $\phi(c) \neq 0$. The proof is complete.

THEOREM 2. *Let A be a connected, commutative normed algebra, over a non-discrete field K with absolute value, and let j be a gaussian element in A. Suppose that the norm of A satisfies a polynomial identity on A (or a polynomial identity of order greater than 1 on a neighbourhood of zero in A). Then K is archimedean, and, if ρ is the exponent for K, there exists a compact space Φ and an isometry σ of A into $C(\Phi; \mathfrak{C}; \rho)$, such that $\sigma(j)$ is the constant function i.*

Proof. Lemma 1 shows that K is archimedean; let ρ be the exponent for K. Let Φ be the set of all homomorphisms ϕ of A into $\mathfrak{C}^{(\rho)}$, such that $\phi(j) = i$ and $||\phi(x)|| \leqslant N(x)$ for all x in A, where N is the norm of A. If ϕ_0 is in Φ, if x is in A, and if ϵ is a positive number, the symbol $\{\phi_0; x; \epsilon\}$ will denote the set of all ϕ in Φ such that $|\phi(x) - \phi_0(x)| < \epsilon$. Then Φ becomes a topological space if the family of all sets of type $\{\phi_0; x; \epsilon\}$ is used as a sub-base for the topology.

For each non-zero x in A, let $\mathscr{D}_x$ be the circular disc $\{\zeta| \; |\zeta|^\rho \leqslant N(x)\}$ in the complex plane. The cartesian product, $\mathscr{D}$, of all the $\mathscr{D}_x$, as x ranges over the non-zero elements of A, is a compact space since every factor is compact. But the space Φ may be identified with a subset of $\mathscr{D}$, if the $\mathscr{D}_x$-co-ordinate of any ϕ in Φ is taken as $\phi(x)$; the relative topology of Φ as a subset of $\mathscr{D}$ is clearly the topology which was given to Φ in the preceding paragraph. It is easily shown that Φ is a closed subset of $\mathscr{D}$, whence Φ is compact since $\mathscr{D}$ is compact.

If x is in A, define $\hat{x}(\phi) = \phi(x)$ for all ϕ in Φ. Then each $\hat{x}$ is a complex-valued function defined on Φ, and each $\hat{x}$ is continuous on Φ because of the kind of topology which was introduced in Φ. The mapping $x \to \hat{x}$ is clearly a homomorphism of A into $C(\Phi; \mathfrak{C}; \rho)$, and $\hat{j}$ is the constant function i. We have $N(x) \geqslant ||\phi(x)|| = |\phi(x)|^\rho = |\hat{x}(\phi)|^\rho$ for all ϕ in Φ, and it follows that $N(x) \geqslant \sup\{|\hat{x}(\phi)|^\rho | \phi \in \Phi\}$ for each x in A. On the other hand, if c is a non-zero element of A, Lemma 1 shows that there is a non-zero homomorphism χ of A into $\mathfrak{C}^{(\rho)}$, with $||\chi(c)|| = N(c)$ and $||\chi(x)|| \leqslant N(x)$ for all x in A. It is easily seen that $\chi(j)$ is a gaussian element of the non-zero ring $\chi(A)$, so that $\chi(j) = i$ or $\chi(j) = -i$. Let $\phi(x)$ equal $\chi(x)$ for all x if $\chi(j) = i$, and let $\phi(x)$ equal the complex-conjugate of $\chi(x)$ for all x if $\chi(j) = -i$. Then $\phi(j) = i$, and ϕ belongs to Φ. Now, $N(c) = ||\chi(c)|| = |\chi(c)|^\rho = |\phi(c)|^\rho = |\hat{c}(\phi)|^\rho$,

so that $N(c) = \sup\{|\hat{c}(\phi)|^{\rho} \mid \phi \in \Phi\}$. The mapping $x \to \hat{x}$ thus preserves norms, and is therefore an isometry since its kernel must be zero.

If the mapping $x \to \hat{x}$ is denoted by σ, then the theorem is established.

THEOREM 3. *Let A be a complete normed algebra which satisfies the hypotheses of Theorem 2, and suppose there is a mapping $x \to x^*$ of A into itself such that $\phi(x^*)$ is the complex-conjugate of $\phi(x)$ whenever x is in A and ϕ is in Φ. Then the mapping σ is an isometry of A onto $C(\Phi; \mathbb{C}; \rho)$, such that $\sigma(x^*)$ is the conjugate function of $\sigma(x)$ for each x in A.*

Proof. Let $\hat{A} = \sigma(A)$, so that $\hat{A}$ is a subring of $C(\Phi; \mathbb{C}; \rho)$.

We note that $\hat{j} = \sigma(j)$ is the constant function i and belongs to $\hat{A}$. Also, if e is the unit element of A, then the mapping $k \to ke$ is an isometry of K into A, and the mapping $k \to \sigma(ke)$ is consequently an isometry of the archimedean field K into $\hat{A}$. Then $\hat{A}$ contains a field isomorphic to the field of rational numbers, and so $\hat{A}$ contains every constant function on Φ which assumes a real, rational value. But $\hat{A}$ is isometric to A and is therefore complete; thus, $\hat{A}$ is uniformly closed in $C(\Phi; \mathbb{C}; \rho)$. It follows that $\hat{A}$ contains every constant real-valued function on Φ. Since $\hat{A}$ is a ring and also contains the constant function i, every constant complex-valued function on Φ belongs to $\hat{A}$. But $\hat{A}$ is closed under multiplication, and $\hat{A}$ may therefore be considered a complex algebra and a subalgebra of the complex algebra $C(\Phi; \mathbb{C}; \rho)$.

Next, we note that if ϕ_1 and ϕ_2 are distinct elements of Φ, then $\phi_1(x) \neq \phi_2(x)$ for some x in A, and so $\hat{x}(\phi_1) \neq \hat{x}(\phi_2)$ for such an x. That is, there is an $\hat{x}$ in $\hat{A}$ which distinguishes ϕ_1 and ϕ_2; in the terminology of **(9)**, $\hat{A}$ is a *separating family* for Φ.

Finally, $[\sigma(x^*)](\phi) = \phi(x^*)$ is the complex-conjugate of $\phi(x) = [\sigma(x)](\phi)$, whenever x is in A and ϕ is in Φ. In other words, $\sigma(x^*)$ is the conjugate function of $\sigma(x)$, for each x in A. Then $\hat{A}$ contains the conjugate function $\sigma(x^*)$ of each of its elements $\sigma(x)$, $\hat{A}$ is a uniformly closed complex subalgebra of $C(\Phi; \mathbb{C}; \rho)$, and $\hat{A}$ is a separating family for Φ. Since $\hat{j}$ is in $\hat{A}$ and vanishes nowhere on Φ, it follows from Corollary 2 of Theorem 10 in **(9)** that $\hat{A}$ coincides with $C(\Phi; \mathbb{C}; \rho)$. Thus, σ is an isometry of A onto $C(\Phi; \mathbb{C}; \rho)$, and this completes the proof.

Theorems 2 and 3 describe the algebra A in terms of its ring structure rather than as an algebra; that is, the isometry σ is a ring isomorphism in those theorems, and not an isomorphism as algebras. This difficulty arises because A is not given as a complex algebra, and also because of the arbitrary selection of the gaussian element j to be carried by σ into the constant function i. However, in the case of a complex algebra with unit e it becomes natural to expect σ to carry $i \cdot e$ into the constant function i; when σ is chosen in such a way σ becomes an isomorphism relative to the structure as complex algebras.

LEMMA 2. *Let A and A' be complex topological algebras, such that A has a*

unit element e, and let ϕ be a continuous homomorphism of A into A' such that $\phi(i\cdot e) = i\cdot\phi(e)$. Then ϕ is a homomorphism of A into A' as algebras over the complex field.

The proof of this lemma is routine, and is left to the reader.

THEOREM 4. *Let ρ be a real number with $0 < \rho \leqslant 1$, and let A be a commutative normed algebra, with unit e, over $\mathfrak{C}^{(\rho)}$. Suppose that the norm of A satisfies a polynomial identity on A (or a polynomial identity of order greater than 1 on a neighbourhood of zero in A). Then there exists a compact space Φ and a norm-preserving isomorphism σ of A into $C(\Phi; \mathfrak{C}; \rho)$, as algebras over $\mathfrak{C}^{(\rho)}$. If, in addition, A is complete and there is a mapping $x \to x^*$ of A into itself such that $\sigma(x^*)$ is the conjugate function of $\sigma(x)$ for every x in A, then σ is an isomorphism of A onto $C(\Phi; \mathfrak{C}; \rho)$.*

Proof. Apply Theorem 2, with $j = i\cdot e$. Then $\sigma(i\cdot e)$ is the constant function i, and σ is continuous since it is an isometry. Lemma 2 may be applied to show that σ is an isomorphism of A into $C(\Phi; \mathfrak{C}; \rho)$ as algebras over $\mathfrak{C}^{(\rho)}$. The remainder of the theorem follows from Theorem 3.

An interesting special case of this theorem merits an explicit statement as a corollary.

COROLLARY. *Let A be a commutative, complete normed algebra with unit e, over $\mathfrak{C}^{(1)}$, and let the norm N for A satisfy the polynomial identity $N(x)^2 = N(x^2)$ on A. Then there exist a compact space Φ and a norm-preserving isomorphism σ of A into $C(\Phi; \mathfrak{C}; 1)$, as algebras over $\mathfrak{C}^{(1)}$. If, in addition, there is a mapping $x \to x^*$ of A into itself such that $\sigma(x^*)$ is the conjugate function of $\sigma(x)$ for each x in A, then σ is a norm-preserving isomorphism of A onto $C(\Phi; \mathfrak{C}; 1)$, as algebras over $\mathfrak{C}^{(1)}$.*

This corollary was first given by Gelfand in (**5**, Satz 16, Folgerung 3), but Φ was defined in a completely different way there. However, the use of Lemma 2 would enable us to show that in this corollary Φ can also be described as the set of all homomorphisms ϕ of A into $\mathfrak{C}^{(1)}$, as algebras over $\mathfrak{C}^{(1)}$, such that $\phi(e) = 1$, and such that $|\phi(x)| \leqslant N(x)$ for all x in A. It would then follow that the kernel of each ϕ in Φ is a maximal ideal in A, and that each maximal ideal in A is the kernel of some ϕ in Φ. Indeed, the mapping which associates with each ϕ the kernel of ϕ is a homeomorphism of Φ onto Gelfand's space of maximal ideals, so that there is really no difference between our corollary and Gelfand's result.

Note 1. In Lemma 1, Theorem 2, and Theorem 3, the only purpose of assuming that the field K is non-discrete is to make it possible to use Theorem 1, in order to show that the norm of the algebra is power multiplicative. Thus, it would be possible to drop the hypothesis that K is non-discrete in these three results, provided that the hypothesis that the norm of the algebra satisfies a polynomial identity is replaced by the assumption that the norm of the algebra is power multiplicative.

Note 2. In Theorems 2 and 3, the assumption that the algebra is connected can be replaced by the assumption that the field K of scalars is archimedean; since the completion of the algebra would be a normed algebra over the completion of K and would therefore be connected because of the connectedness of the completion of K, the conclusions of these theorems would apply to the completion of the algebra and therefore to the original algebra.

References

1. R. Arens, *Linear topological division algebras*, Bull. Amer. Math. Soc., *53* (1947), 623–630.
2. S. Aurora, *Multiplicative norms for metric rings*, Pacific J. Math., *7* (1957), 1279–1304.
3. ——— *On power multiplicative norms*, Amer. J. Math., *80* (1958), 879–894.
4. N. Bourbaki, Éléments de mathématique, Livre III. *Topologie générale*, Actualités Scientifiques et Industrielles, nos. 858, 916, 1029, 1045, 1084 (Paris, 1940–1951).
5. I. Gelfand, *Normierte Ringe*, Rec. Math. (Mat. Sb.) N.S., *9* (1941), 3–24.
6. R. Kadison, *A representation theory for commutative topological algebra*, Mem. Amer. Math. Soc., *7* (New York, 1951).
7. S. Mazur, *Sur les anneaux linéaires*, C.R. Acad. Sci. Paris, *207* (1938), 1025–1027.
8. A. Ostrowski, *Ueber einige Lösungen der Funktionalgleichung* $\phi(x)\cdot\phi(y) = \phi(xy)$, Acta Math., *41* (1918), 271–284.
9. M. H. Stone, *The generalized Weierstrass approximation theorem*, Math. Mag., *21* (1948), 167–184, 237–254.
10. L. Tornheim, *Normed fields over the real and complex fields*, Mich. Math. J., *1* (1952), 61–68.

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ON NORMED ALGEBRAS WHICH SATISFY A REALITY CONDITION

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1. Introduction. Various results exist which permit real Banach algebras satisfying some sort of "reality condition" to be identified with the algebra of all continuous real-valued functions on a suitable compact space (or with the algebra of all continuous real-valued functions that "vanish at infinity" on a suitable non-compact, locally compact space in case the algebra has no unit). (Terminology follows **(4)**, so that compact and locally compact spaces must be Hausdorff spaces, in addition to satisfying the usual requirements.) Kadison established such a result in **(6**, Theorem 6.6**)** for any Banach algebra with unit, if the algebra satisfies an appropriate reality condition and has a norm N such that $N(x)^2 = N(x^2)$ for all x. Another such theorem was obtained by Segal in **(8**, Theorem 1**)**, for commutative Banach algebras with unit, if the norm N satisfies the conditions $N(u^2) = N(u)^2$ and $N(u^2 - v^2) \leqslant \max(N(u^2), N(v^2))$ for all u and v; here the second condition imposed upon the norm serves as a reality condition.

In this note, we shall consider commutative, connected, complete normed algebras, over a non-discrete field with absolute value, such that the norm of the algebra *satisfies a polynomial identity* on the algebra in the sense of **(3)**. The algebras are also subjected to a fairly strong reality condition: *it is assumed that whenever I is a closed two-sided ideal such that xy in I implies x in I or y in I, then $x^2 + y^2$ in I implies that x and y are in I.* With these restrictions, it is shown in Theorem 2 that such an algebra is algebraically and topologically isomorphic to the ring of all continuous real-valued functions on a suitable compact space if the algebra has a unit, and to the ring of all continuous real-valued functions which "vanish at infinity" on a suitable non-compact, locally compact space if the algebra is without unit.

The method of proof used in this note is similar to that in **(3)**.

2. Preliminaries. Familiarity with the contents of **(2)** and of **(3)** is presupposed in this note.

If ρ is a real number such that $0 < \rho \leqslant 1$, then the field of all real numbers, normed with the ρth power of the ordinary absolute value, will be denoted by the symbol $\Re^{(\rho)}$. Clearly, $\Re^{(\rho)}$ is a connected field with absolute value and is complete.

Our definitions of compact and locally compact spaces will agree with those in **(4**, chapter 1**)**; that is, a *compact* space is a Hausdorff space for which

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every open covering has a finite subcovering, and a *locally compact* space is a Hausdorff space such that each point is interior to at least one compact subspace.

If Φ is a compact space and ρ is a real number with $0 < \rho \leqslant 1$, the symbol $C(\Phi; \Re; \rho)$ will denote the set of all continuous real-valued functions $x(\phi)$ on Φ, with algebraic operations defined in the obvious way, and with the norm N such that $N(x) = \sup\{|x(\phi)|^\rho | \phi \in \Phi\}$ for all x in $C(\Phi; \Re; \rho)$. In case Φ is a locally compact space which is not compact, then $C(\Phi; \Re; \rho)$ will be the set of all continuous real-valued functions on Φ which "vanish at infinity," if ρ is a real number with $0 < \rho \leqslant 1$; the definitions for the algebraic operations and for the norm remain the same as before. Then, whenever Φ is a locally compact space and ρ is a real number with $0 < \rho \leqslant 1$, the system $C(\Phi; \Re; \rho)$ is defined and is a connected, commutative, complete normed algebra over $\Re^{(\rho)}$. The norm defined on $C(\Phi; \Re; \rho)$ is simply the ρth power of the usual supremum norm, and it follows that the topology in $C(\Phi; \Re; \rho)$ is the topology of uniform convergence on Φ. It may also be noted that the norm of $C(\Phi; \Re; \rho)$ is *power multiplicative*. (See **(2)** for a definition of that concept.)

If Φ is a locally compact space, it is possible to construct a compact space Ψ which contains every point of Φ and precisely one point not in Φ. In case Φ is locally compact but not compact, this is achieved by the standard "one point compactification" (see **(4**, chapter I, § 10, Theorem 3**)**, for instance), in which a "point at infinity" ω is added to Φ to produce the compact space Ψ. The case of a compact space Φ is handled by adjoining an isolated point ω to Φ in order to form Ψ. In either case, a compact space Ψ is obtained by the addition of a single point ω to Φ.

Conversely, if a point ω is selected in a compact space Ψ, then the complement, Φ, of ω in Ψ is open and hence locally compact. (See **(4**, chapter I, § 10, Proposition 10**)**.) Thus, there exists a one-to-one correspondence between locally compact spaces Φ, and pairs consisting of a compact space Ψ and a selected point ω in Ψ. (The correspondence is actually between equivalence classes, where the equivalence relation is homeomorphism.) This correspondence is related in an interesting way to the algebras of continuous functions which were introduced above. Let ρ be a real number with $0 < \rho \leqslant 1$, let Ψ be a compact space with a selected point ω in Ψ, and let $\hat{A}$ be the subalgebra of $C(\Psi; \Re; \rho)$ consisting of those functions in $C(\Psi; \Re; \rho)$ which vanish at ω. If Φ is the complement of ω in Ψ, and if τ is the mapping which carries every function in $\hat{A}$ into its restriction to Φ, then it is easily verified that τ is a norm-preserving isomorphism of $\hat{A}$ onto $C(\Phi; \Re; \rho)$, as algebras over $\Re^{(\rho)}$.

The principal result of this section will now be proved; this result will show that if a complete normed algebra over an $\Re^{(\rho)}$ has sufficiently many homomorphisms into $\Re^{(\rho)}$, as algebras over $\Re^{(\rho)}$, then the algebra is a $C(\Phi; \Re; \rho)$ for a suitable locally compact space Φ. This theorem will be used in the next section in proving the principal theorem.

THEOREM 1. *Let A be a non-zero, complete normed algebra over $\Re^{(\rho)}$, for a fixed real number ρ such that $0 < \rho \leqslant 1$. Suppose that for every non-zero c in A there is a homomorphism ϕ of A into $\Re^{(\rho)}$, as algebras over $\Re^{(\rho)}$, such that $||\phi(x)|| \leqslant N(x)$ for all x in A, and $||\phi(c)|| = N(c)$, where N is the norm of A. Then there exists a locally compact space Φ and a norm-preserving isomorphism σ of A onto $C(\Phi; \Re; \rho)$, as algebras over $\Re^{(\rho)}$, such that Φ is compact if and only if A has a unit element.*

Proof. Let Ψ be the set of all homomorphisms ϕ of A into $\Re^{(\rho)}$, as algebras over $\Re^{(\rho)}$, such that $||\phi(x)|| \leqslant N(x)$ for all x in A. The zero homomorphism of A into $\Re^{(\rho)}$ will be denoted by ω, and clearly ω belongs to Ψ; the complement of ω in Ψ will be written Φ.

If ϕ_0 is in Ψ, if $x_1, \ldots, x_n$ are elements of A, and if ϵ is a positive number, then the symbol $\{\phi_0; x_1, \ldots, x_n; \epsilon\}$ will represent the set of all ϕ in Ψ such that $|\phi(x_i) - \phi_0(x_i)| < \epsilon$ for $i = 1, \ldots, n$. Let $\mathscr{U}$ be the family of all sets $\{\phi_0; x_1, \ldots, x_n; \epsilon\}$ as ϕ_0 ranges over the elements of Ψ, as $\{x_1, \ldots, x_n\}$ ranges over all finite subsets of A, and as ϵ ranges over all positive numbers. Then if $\mathscr{U}$ is used as a base for open sets in Ψ we obtain a topological space Ψ.

Now, let $\mathscr{E}_x$ be the closed interval $\{t \mid |t|^\rho \leqslant N(x)\}$, for each non-zero x in A. If $\mathscr{E}$ is the cartesian product of the $\mathscr{E}_x$ as x ranges over all non-zero elements of A, then $\mathscr{E}$ is compact since each $\mathscr{E}_x$ is compact. But Ψ may be identified with a subset of $\mathscr{E}$ in an obvious way, by letting the $\mathscr{E}_x$-co-ordinate of ϕ be $\phi(x)$, if ϕ is in Ψ. Then the topology which was previously defined in Ψ is clearly the relative topology of Ψ as a subset of $\mathscr{E}$. It is easily shown that Ψ is a closed set in $\mathscr{E}$, so that Ψ is compact.

For x in A, define $\hat{x}(\phi) = \phi(x)$ for all ϕ in Ψ, so that $\hat{x}$ is a real-valued function defined on Ψ. Because of the nature of the topology which was introduced in Ψ, each $\hat{x}$ must be continuous on Ψ. The mapping $x \to \hat{x}$ then becomes a homomorphism of A into $C(\Psi; \Re; \rho)$, as algebras over $\Re^{(\rho)}$.

If c is a non-zero element of A, then $N(c) \geqslant ||\phi(c)|| = |\phi(c)|^\rho = |\hat{c}(\phi)|^\rho$ for all ϕ in Ψ, and it follows that $N(c) \geqslant \sup\{|\hat{c}(\phi)|^\rho \mid \phi \in \Psi\}$. On the other hand, the hypothesis of this theorem asserts the existence of a ϕ in Ψ such that $N(c) = ||\phi(c)|| = |\phi(c)|^\rho = |\hat{c}(\phi)|^\rho$, so that $N(c) = \sup\{|\hat{c}(\phi)|^\rho \mid \phi \in \Psi\}$. Thus, the mapping $x \to \hat{x}$ is norm-preserving, and is consequently an isomorphism since its kernel would have to be zero.

Let $\hat{A}$ be the image in $C(\Psi; \Re; \rho)$ of the mapping $x \to \hat{x}$, so that $\hat{A}$ is a subalgebra of $C(\Psi; \Re; \rho)$. Since A is complete and is isometric to $\hat{A}$, it follows that $\hat{A}$ is also complete, and $\hat{A}$ must therefore be uniformly closed in $C(\Psi; \Re; \rho)$. Also, each element of $\hat{A}$ clearly vanishes at ω. Finally, if ϕ_1 and ϕ_2 are distinct elements of Ψ, then $\phi_1(x) \neq \phi_2(x)$ for some x in A, whence $\hat{x}(\phi_1) \neq \hat{x}(\phi_2)$ for such an x. That is, every pair of distinct points of Ψ may be distinguished by an element of $\hat{A}$; in the language of **(9)**, $\hat{A}$ is a *separating family* for Ψ. Corollary 2 of Theorem 5 in **(9)** then shows that $\hat{A}$ is precisely the subalgebra of all elements of $C(\Psi; \Re; \rho)$ which vanish at ω.

But the discussion preceding this theorem then shows that there is a norm-preserving isomorphism τ of $\hat{A}$ onto $C(\Phi; \Re; \rho)$, as algebras over $\Re^{(\rho)}$. Let σ be the mapping obtained by following the mapping $x \to \hat{x}$ by the mapping τ. Then σ is a norm-preserving isomorphism of A onto $C(\Phi; \Re; \rho)$, as algebras over $\Re^{(\rho)}$.

In case A has a unit element e, then $\{\omega; e; 1/2\}$ is an open set in Ψ and contains only ω, so that Φ is closed in Ψ, and Φ is therefore compact. Conversely, if Φ is compact, then the constant function 1 belongs to $C(\Phi; \Re; \rho)$ and acts as a unit element there; thus, A must have a unit element also, for A is isomorphic to $C(\Phi; \Re; \rho)$. This completes the proof.

Note. In Theorem 1, the space Φ consists of all non-zero homomorphisms ϕ of A into $\Re^{(\rho)}$, as algebras over $\Re^{(\rho)}$, such that $||\phi(x)|| \leqslant N(x)$ for all x in A. However, if ϕ is a non-zero homomorphism of A into $\Re^{(\rho)}$ such that $||\phi(x)|| \leqslant N(x)$ for all x in A, then the inequality $||\phi(y) - \phi(z)|| = ||\phi(y - z)|| \leqslant N(y - z)$ shows that ϕ is continuous, and it follows easily that ϕ is a homomorphism of A into $\Re^{(\rho)}$,as algebras over $\Re^{(\rho)}$. Thus, the space Φ can be described as the set of all non-zero homomorphisms ϕ of A into $\Re^{(\rho)}$, such that $||\phi(x)|| \leqslant N(x)$ for all x in A. (That is, the elements of Φ may be regarded either as algebra-homomorphisms or as ring-homomorphisms.)

3. Normed algebras which satisfy a reality condition. A two-sided ideal I in a ring R will be called a *prime* ideal if xy in I implies x in I or y in I; a two-sided ideal I in a ring R will be called a *real* ideal if $x^2 + y^2$ in I implies that x and y are in I. We shall say that a topological ring R *satisfies the reality condition* if every closed prime ideal in R is a real ideal. Kadison has considered a weaker type of reality condition in **(6)**, where "strictly real" algebras are considered; the present methods seem to require a stronger reality condition, however, such as the one we have just given.

In this section, Theorem 1 will be used in order to obtain a representation theorem for connected, commutative, complete normed algebras which satisfy the reality condition and which have a norm that satisfies a polynomial identity in the sense of **(3)**. It will be shown that such an algebra can be identified with a $C(\Phi; \Re; \rho)$, for some locally compact space Φ and for some real number ρ with $0 < \rho \leqslant 1$ .First, some useful lemmas are obtained.

LEMMA 1. *Let E be an archimedean field with absolute value, such that E is a normed algebra over a complete field F with absolute value. Then E is complete.*

The proof is routine, and is left to the reader.

LEMMA 2. *Let A be a connected normed algebra over a complete field F with absolute value, and let E be a field with absolute value such that E is a normed algebra over F. Suppose that there is a continuous non-zero homomorphism ϕ of A into E, as algebras over F, and suppose that the zero ideal is a real ideal in E. Then there is a real number ρ with $0 < \rho \leqslant 1$, such that there exist unique isometries of E and of F onto $\Re^{(\rho)}$.*

Proof. $\phi(A)$ is the continuous image of the connected set A, and is therefore connected in E. Since ϕ is not the zero homomorphism, $\phi(A)$ is a connected set which contains a non-zero element in addition to zero. The field E is not totally disconnected, then, and Lemma 15 of **(1)** implies that E is archimedean. Lemma 1 shows that E is also complete.

Let ρ be the exponent for E, as defined in § 3 of **(3)**. Then there is an isometry σ of E into $\mathfrak{C}^{(\rho)}$. Because E is complete, σ is actually an isometry of E onto $\mathfrak{R}^{(\rho)}$ or $\mathfrak{C}^{(\rho)}$. But the zero ideal is not a real ideal in $\mathfrak{C}^{(\rho)}$, so that σ is an isometry of E onto $\mathfrak{R}^{(\rho)}$. There is an obvious isometry of F into E, and F is therefore isometric to a complete subfield of $\mathfrak{R}^{(\rho)}$, whence F is isometric to $\mathfrak{R}^{(\rho)}$.

Finally, if L is any field with absolute value such that L is isometric to some $\mathfrak{R}^{(\rho)}$, then there is precisely one isometry of L onto $\mathfrak{R}^{(\rho)}$. For, all isometries of L onto $\mathfrak{R}^{(\rho)}$ must agree on the prime field of L and therefore on the closure of the prime field of L. But L is the closure of its prime field, since L is isometric to $\mathfrak{R}^{(\rho)}$, which is the closure of its prime field. Thus, all isometries of L onto $\mathfrak{R}^{(\rho)}$ must coincide. In particular, the isometries of E and of F onto $\mathfrak{R}^{(\rho)}$ are uniquely determined.

THEOREM 2. *Let A be a non-zero, connected, commutative, complete normed algebra, over a non-discrete field K with absolute value. Suppose that A satisfies the reality condition, and suppose that the norm of A satisfies a polynomial identity on A (or a polynomial identity of order greater than 1 on a neighbourhood of zero in A). Then there exist a real number ρ, with $0 < \rho \leqslant 1$, and a locally compact space Φ, such that there is an isometry σ of A onto $C(\Phi; \mathfrak{R}; \rho)$, and such that Φ is compact and if only if A has a unit element.*

Proof. If F is the completion of K, then **(4**, chapter IX, § 3, no. 7**)** indicates that the completion of A is a normed algebra over F, so that A may be regarded as a normed algebra over F since A is complete.

The norm, N, of A is power multiplicative, by Theorem 1 of **(3)**. If c is a non-zero element of A, let $\mathcal{N}$ be the set of all power multiplicative pseudo-norms N' subordinate to N and such that: (i) $N'(c) = N(c)$, (ii) $N'(cx) = N'(c) \cdot N'(x)$ for all x in A, and (iii) $N'(kx) = ||k|| \cdot N'(x)$ whenever k is in F and x is in A. It is easily verified that $\mathcal{N}$ contains N_c and is a *hereditary system*, in the terminology of **(2)**. As in Lemma 4 of **(2)**, it can be shown that $\mathcal{N}$ contains a minimal element N', and **(2**, Lemma 2**)** shows that N' is a pseudo absolute value.

The null ideal $I(N')$ is an ideal and a subalgebra of A, so that $A/I(N')$ is an algebra over F. Let $\bar{N}$ be the function defined on $A/I(N')$ such that the value $\bar{N}(X)$ assumed by $\bar{N}$ on the residue class X (modulo $I(N')$) is the same as the constant value assumed by N' on the elements of X. Then the algebra $\bar{A} = A/I(N')$, with the function $\bar{N}$ as its norm, becomes a commutative normed algebra over F, and $\bar{N}$ is an absolute value on $\bar{A}$. If η is the natural mapping of A onto $\bar{A}$, then η is a homomorphism of A onto $\bar{A}$, as algebras

over F, and $\bar{N}(\eta(x)) = N'(x) \leqslant N(x)$ for all x in A, while $\bar{N}(\eta(c)) = N'(c) = N(c) \neq 0$. The fact that $\bar{N}(\eta(c)) \neq 0$ shows that $\bar{A}$ contains a non-zero element, $\eta(c)$. Thus, $\bar{A}$ is a non-zero commutative algebra and has no proper zero-divisors since $\bar{N}$ is an absolute value.

A standard procedure permits the formation of a field E of quotients for $\bar{A}$, and E also is an algebra over F. If $a, s \in \bar{A}$, with $s \neq 0$, let $||[a/s]|| = \bar{N}(a)/\bar{N}(s)$ define a norm in E. Then E becomes a field with absolute value such that E is a normed algebra over F. The natural mapping ξ of $\bar{A}$ into E is clearly a norm-preserving isomorphism of $\bar{A}$ into E, as algebras over F. Let ϕ be the mapping obtained by applying first η, and then ξ. It is evident that ϕ is a homomorphism of A into E, as algebras over F. Also, $||\phi(c)|| = ||\xi(\eta(c))|| = \bar{N}(\eta(c)) = N'(c) = N(c) \neq 0$, so that $\phi(c) \neq 0$, whence ϕ is a non-zero homomorphism. For x and y in A, we have $||\phi(x) - \phi(y)|| = ||\phi(x - y)|| = ||\xi(\eta(x - y))|| = \bar{N}(\eta(x - y)) = N'(x - y) \leqslant N(x - y))$, and this inequality shows that ϕ is continuous.

Lemma 5 of **(2)** shows that the subordinate pseudonorm N' of A must be continuous on A. Thus, $I(N')$ must be a closed set since it is the pre-image under N' of the closed set 0. Also, $I(N')$ is a prime ideal since N' is a pseudo absolute value. But A satisfies the reality condition, and the closed prime ideal $I(N')$ is therefore a real ideal in A. It follows easily that the zero ideal of $\bar{A}$ is a real ideal; that is, if X and Y in $\bar{A}$ are such that $X^2 + Y^2 = 0$, then $X = Y = 0$. Then E also has the property that if X and Y in E are such that $X^2 + Y^2 = 0$, then $X = Y = 0$. Thus, the zero ideal of E is a real ideal.

Lemma 2 may now be applied, so that there are unique isometries of E and of F onto $\mathfrak{R}^{(\rho)}$, for some real number ρ with $0 < \rho \leqslant 1$. We note that ρ must be the exponent for F, and ρ is consequently also the exponent for K; thus, ρ is determined by K. We shall now identify E and F with $\mathfrak{R}^{(\rho)}$, so that A is a non-zero, complete normed algebra over $\mathfrak{R}^{(\rho)}$; and for each non-zero c in A we have a homomorphism ϕ of A into $\mathfrak{R}^{(\rho)}$, as algebras over $\mathfrak{R}^{(\rho)}$, such that $||\phi(c)|| = N(c)$, as we have shown above. Finally, each ϕ also is such that $||\phi(x)|| = ||\xi(\eta(x))|| = \bar{N}(\eta(x)) = N'(x) \leqslant N(x)$ for all x in A, and A therefore satisfies the hypotheses of Theorem 1. It follows that there exist a locally compact space Φ and a norm-preserving isomorphism σ of A onto $C(\Phi; \mathfrak{R}; \rho)$, as algebras over $\mathfrak{R}^{(\rho)}$, such that Φ is compact if and only if A has a unit element. Since A was not given originally as an algebra over $\mathfrak{R}^{(\rho)}$, it is preferable to describe σ as a ring-isomorphism rather than an algebra-isomorphism in the statement of the present theorem.

It seems desirable to describe the space Φ in terms of the normed algebra as it is originally given in Theorem 2. The note at the end of § 2 permits us to describe Φ as the set of all non-zero homomorphisms ϕ of A into $\mathfrak{R}^{(\rho)}$ (as rings), such that $||\phi(x)|| \leqslant N(x)$ for all x in A, where ρ is the exponent for K.

The question might be raised whether any of the hypotheses in Theorem 2

may be dropped if the existence of an isometry of A *into* a $C(\Phi; \Re; \rho)$ is regarded as a satisfactory conclusion.

First, the reality condition cannot be dropped if we desire such a conclusion. For, the field $\mathfrak{C}$ of complex numbers satisfies every hypothesis of Theorem 2, except for the reality condition; but $\mathfrak{C}$ contains an element i, the negative of whose square is a non-zero idempotent, while no $C(\Phi; \Re; \rho)$ can contain such an element. Thus, $\mathfrak{C}$ can never be isometric to a subring of a $C(\Phi; \Re; \rho)$.

Connectedness of A is also essential in Theorem 2. For the field P_3 of 3-adic numbers (see example 9 in **(1)**, for instance) is a normed algebra over itself and satisfies all hypotheses of the theorem other than connectedness; but P_3 is non-archimedean and therefore cannot be isometric to a subring of a $C(\Phi; \Re; \rho)$, since every such subring is archimedean if it is not zero.

Completeness of A cannot be dropped as a hypothesis of Theorem 2, either, for Dieudonné constructed in **(5)** a dense, connected subfield F of $\mathfrak{C}$, such that F is algebraically isomorphic to the field of real numbers; such a field, as a normed algebra over itself, with the ordinary absolute value as the norm, satisfies every hypothesis of the theorem excepting completeness. But if there were an isometry of F into a $C(\Phi; \Re; \rho)$, completion would yield an isometry of $\mathfrak{C}$, the completion of F, into $C(\Phi; \Re; \rho)$, and this is impossible. Thus, F satisfies all conditions of Theorem 2 except for completeness, and F cannot be isometric to a subring of a $C(\Phi; \Re; \rho)$.

Note 1. The assumption that K is non-discrete is used in Theorem 2 only to show that the norm of the algebra is power multiplicative, so that this theorem remains true if the assumption that K is non-discrete is dropped, provided that the hypothesis that the norm satisfies a polynomial identity is replaced by the stronger assumption that the norm is power multiplicative.

Note 2. The hypothesis of connectedness can be replaced in Theorem 2 by the assumption that the field K is archimedean, for the completion of K would then be connected, and it would follow that A is connected since it is a normed algebra over the completion of K.

References

1. S. Aurora, *Multiplicative norms for metric rings*, Pacific J. Math., *7* (1957), 1279–1304.
2. ——— *On power multiplicative norms*, Amer. J. Math., *80* (1958), 879–894.
3. ——— *On normed algebras whose norms satisfy polynomial identities*, Can. J. Math., *13* (1961), 000–000.
4. N. Bourbaki, Éléments de mathématique, Livre III. *Topologie générale*, Actualités Scientifiques et Industrielles, nos. 858, 916, 1029, 1045, 1084 (Paris, 1940–1951).
5. J. Dieudonné, *Sur les corps topologiques connexes*, C. R. Acad. Sci. Paris, *221* (1945), 396–398.

6. R. Kadison, *A representation theory for commutative topological algebra*, Mem. Amer. Math. Soc., 7 (New York, 1951).
7. A. Ostrowski, *Ueber einige Lösungen der Funktionalgleichung* $\phi(x)\cdot\phi(y) = \phi(xy)$, Acta Math., *41* (1918), 271–284.
8. I. E. Segal, *Postulates for general quantum mechanics*, Ann. Math., *48* (1947), 930–948.
9. M. H. Stone, *The generalized Weierstrass approximation theorem*, Math. Mag., *21* (1948), 167–184, 237–254.

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COUNTING COLOURED GRAPHS

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1. Introduction. A *graph on n labelled nodes* is a set of n objects called "nodes," distinguishable from each other, and a set (possibly empty) of "edges," that is, pairs of nodes. Each edge is said to join its pair of nodes, at most one edge joins any two nodes and no edge joins a node to itself. By a k-colouring of such a graph we mean a mapping of the nodes of the graph onto a set of k distinct colours, such that no two nodes joined by an edge are mapped onto the same colour. We take $k > 1$.

Following Read **(1)**, we write $M_n = M_n(k)$ for the number of such coloured graphs, $F_n = F_n(k)$ for the number of such coloured graphs in which there is at least one node mapped onto each colour, and $f_n = f_n(k)$ for the number of those graphs of the latter set which are connected. We write also $T(\alpha) = 2^{\alpha}$ and use $\sum$ to denote summation over all i such that $1 \leqslant i \leqslant k$. Read **(1)** showed that

$$M_n(k) = \sum_{(n)} \frac{n!}{s_1! \dots s_k!}\, T\left(\frac{1}{2}n^2 - \frac{1}{2}\sum s_i^2\right), \tag{1.1}$$

where $\sum_{(n)}$ denotes summation over all sets of non-negative integers s_i such that

$$\sum s_i = n. \tag{1.2}$$

$F_n(k)$ is the corresponding sum in which every s_i is positive. Read also shows that

$$f_n(k) = F_n(k) - \sum_{r=1}^{n-1} \binom{n-1}{r-1} F_{n-r}(k) f_r(k), \tag{1.3}$$

where $F_1(k) = F_2(k) = \dots = F_{k-1}(k) = 0$.

If we put

$$\psi = \psi(x) = \sum_{r=1}^{\infty} T(-\tfrac{1}{2}r^2)\frac{x^r}{r!},$$

we have

$$\psi^k = \sum_{n=k}^{\infty} \frac{T(-\tfrac{1}{2}n^2)F_n(k)x^n}{n!} \tag{1.4}$$

and

$$(1+\psi)^k = 1 + \sum_{n=1}^{\infty} \frac{T(-\tfrac{1}{2}n^2)M_n(k)x^n}{n!}, \tag{1.5}$$

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as remarked by Read. Hence, or directly from (1.1), we have

$$M_n(k) = \sum_{r=1}^{k} \binom{k}{r} F_n(r). \tag{1.6}$$

The series for ψ and the series in (1.4) and (1.5) are convergent for all x and so represent integral functions. On the other hand, Read deduces (1.3) from the formal relationship

$$1 + \sum_{n=k}^{\infty} \frac{F_n(k)x^n}{n!} = \exp\left(\sum_{n=1}^{\infty} \frac{f_n(k)x^n}{n!}\right),$$

in which both series diverge for all non-zero values of x.

Read deduces from (1.4) and (1.5) respectively that

$$F_n(k) = \sum_{r=1}^{n-1} \binom{n}{r} 2^{r(n-r)} F_r(k-1) \tag{1.7}$$

and

$$M_n(k) - M_n(k-1) = \sum_{r=0}^{n-1} \binom{n}{r} 2^{r(n-r)} M_r(k-1) \tag{1.8}$$

and uses these to compute F_n and M_n for small values of k and n. He remarks that M_n, unlike F_n, is a polynomial in k of degree n and this follows from (1.8) by induction. He uses (1.8) to calculate M_n as a polynomial in k for $n = 1, 2, 3, 4$.

If we differentiate (1.5) logarithmically with respect to x, rearrange and equate coefficients of x^n, we have

$$M_n(k) = \sum_{r=1}^{n-1} 2^{r(n-r)} M_r(k) \left\{ \binom{n-1}{r} k - \binom{n-1}{r-1} \right\} + k \tag{1.9}$$

and this expresses $M_n(k)$ in terms of $M_r(k)$ for $r < n$ and does not involve $M_n(k-1)$. The polynomial property of $M_n(k)$ follows from (1.9) by induction even more trivially than from (1.8). Again, if we put

$$M_n(k) = \sum_{s=1}^{n} a_{ns} k^s,$$

substitute in (1.9), and equate the coefficients of powers of k, we find the recurrence formula

$$a_{ns} = \sum_{r=s-1}^{n-1} 2^{r(n-r)} \binom{n-1}{r} a_{r,s-1} - \sum_{r=s}^{n-1} 2^{r(n-r)} \binom{n-1}{r-1} a_{rs}$$

for the coefficients in the polynomial $M_n(k)$ when $s > 1$. In particular

$$a_{nn} = 2^{n-1} a_{n-1,n-1} = \ldots = 2^{\frac{1}{2}n(n-1)}.$$

Similarly we can obtain

$$(n-k)F_n(k) = \sum_{s=k}^{n-1} 2^{(s-1)(n-s)} F_s(k) \left\{ \binom{n}{s} k - \binom{n}{s-1} \right\}.$$

Although $F_n(k)$ is not a polynomial in k, we have

$$F_n(k) = \frac{n!}{(n-k)!}\, 2^{kn-\frac{1}{2}k^2-\frac{1}{2}k} J_{n-k}(k),$$

where $J_n(k)$ is a polynomial of the nth degree in k such that $J_0 = 1$ and, for $n \geqslant 1$,

$$n(n+1)J_n = \sum_{u=0}^{n-1} 2^{(u-1)(n-u)} \binom{n+1}{u} \{(n-u)k - u\} J_u.$$

2. The main theorems. But these results are fairly trivial. Our purpose here is to find asymptotic formulae for the behaviour of M_n, F_n, and f_n for fixed k as $n \to \infty$. We define a as the least positive residue of n to modulus k. We use A (with or without a suffix) to denote a positive number, not necessarily the same at each occurrence, which depends at most on k and on its suffix, if any. The notation $O(\ )$ refers to the passage of n to infinity and the positive number involved is A_H. We write $K = \frac{1}{2}\{1 - (1/k)\}$.

We shall prove

THEOREM 1. *As $n \to \infty$, we have*

$$M_n = \left(\frac{k}{n \log 2}\right)^{\frac{1}{2}(k-1)} k^n 2^{Kn^2} \left\{ \sum_{h=0}^{H-1} C_h n^{-h} + O(n^{-H}) \right\}, \tag{2.1}$$

where $C_h = C_h(k, a)$ depends on k, h and the residue of n (mod k), but not otherwise on n.

THEOREM 2. *As $n \to \infty$,*

$$F_n \sim f_n \sim M_n.$$

In fact,

$$F_n/M_n = 1 + O(e^{-An^2}), \qquad f_n/M_n = 1 + O(e^{-An})$$

and so (2.1) *remains true, with unaltered coefficients $C_0, C_1, \ldots$, if M_n is replaced by F_n or by f_n.*

Theorem 2, which we deduce fairly simply from Theorem 1, disposes of F_n and f_n. The coefficients C_h are of interest. Each can be expressed in terms of one or more multiple series. In particular,

$$C_0(k, a) = k^{\frac{1}{2}} (\log 2)^{\frac{1}{2}(k-1)} (2\pi)^{-\frac{1}{2}(k-1)} L(a), \tag{2.2}$$

where

$$L(a) = \sum_{((a))} T\left(-\frac{1}{2}\sum s_i^2 + \frac{a^2}{2k}\right) \tag{2.3}$$

and the sum $\sum_{((a))}$ is over all integral values of the s_i, positive, negative, or zero, subject to the condition

$$\sum s_i = a. \tag{2.4}$$

It can be shown very simply that $L(-a) = L(a)$ and that $L(k+a) = L(a)$. Apart from a trivial factor, $L(a)$ is a generalized theta-function and the transformation theory of such a function may be used to obtain information about the value of $C_0(k, a)$. This involves a good deal of elaborate detail, however, and we postpone it to a sequel. Here we prove more simply that $C_0(k, a)$ differs from 1 by a very small amount.

THEOREM 3. *If* $\epsilon = 1.33 \times 10^{-6}$, *then*

$$1 - \epsilon < C_0(k, a) < 1 + \epsilon$$

for $k \leqslant 1000$. *For* $k > 1000$, *we have*

$$(1 - \epsilon)(1 - 10^{-12})^k < C_0(k, a) < (1 + \epsilon)(1 + 10^{-12})^k.$$

But $C_0(k, a)$ is not independent of a. In fact, we shall show that

$$C_0(2, 0) \neq C_0(2, 1). \tag{2.5}$$

To put it roughly, $C_0(2, a)$ does depend on a, though only very little.

It is not surprising that M_n (and F_n and f_n), like other enumerative functions, should depend on the residue of $n \pmod k$ as well as on the size of n. For example, the number of partitions of n into k parts can be expressed as a sum of powers of n up to n^{k-1}, the coefficients of which depend on the residue of $n \pmod{k!}$. But the coefficients of the larger powers, in particular n^{k-1}, do not depend on this residue. Again, the well-known "singular series" in Waring's Problem depends on the arithmetical properties of n. But these enumerative functions of n are fairly small. The asymptotic expansions of the larger enumerative functions (for example, $p(n)$, the number of partitions of n into any number of parts, for which $p(n) \sim B_0 n^{-1} \exp(B_1 \sqrt{n})$) do not have the coefficients of their dominant terms dependent on the congruence properties of n. Thus it is a somewhat unusual phenomenon that M_n, which is very large indeed, has C_0 depending on the residue of $n \pmod k$ but, according to Theorem 3, only a little. The distinction between the size of n and its arithmetical properties, and indeed the whole of the remarks of this paragraph, are deliberately vague. But the point involved seems in some ways the most interesting part of the results.

3. Proof of Theorem 2. If we write $u_i = ks_i - n$ and

$$s_0 = 1, \quad S_m = \sum u_i^m = \sum (ks_i - n)^m \quad (m > 0)$$

and suppose (1.2) to be satisfied we have

$$S_1 = \sum (ks_i - n) = k(\sum s_i - n) = 0, \tag{3.1}$$

$$S_2 = k^2 \sum s_i^2 - 2kn \sum s_i + kn^2 = k^2 \sum s_i^2 - kn^2 \tag{3.2}$$

and so

$$k^2(n^2 - \sum s_i^2) = k(k-1)n^2 - S_2 = 2k^2 K n^2 - S_2. \tag{3.3}$$

Since $S_2 \geqslant 0$, we have

$$T(\tfrac{1}{2}n^2 - \tfrac{1}{2}\sum s_i^2) = T(Kn^2 - \tfrac{1}{2}k^{-2}S_2) \leqslant T(Kn^2).$$

and, by (1.1),

$$M_n(k) \leqslant T(Kn^2) \sum_{(n)} \frac{n!}{s_1! \dots s_k!} = k^n T(Kn^2).$$

This is true for all k and all n.

It follows from the definitions that

$$0 \leqslant f_n(k) < F_n(k) \leqslant M_n(k)$$

for all $n > k$. Again, by (1.6),

$$\begin{aligned} M_n(k) - F_n(k) &= \sum_{r=1}^{k-1} \binom{k}{r} F_n(r) \leqslant \sum_{r=1}^{k-1} \binom{k}{r} M_n(r) \\ &\leqslant (k-1)^n T\left\{\frac{1}{2}n^2\left(1 - \frac{1}{k-1}\right)\right\} \sum_{r=1}^{k-1} \binom{k}{r} \\ &\leqslant 2^k (k-1)^n T\{Kn^2 - \tfrac{1}{2}n^2/(k^2-k)\}. \end{aligned}$$

If we now assume Theorem 1 to be true, we have

$$\begin{aligned} (3.4) \qquad (M_n - F_n)/M_n &< An^{\frac{1}{2}(k-1)}\{1 - (1/k)\}^n T\{-\tfrac{1}{2}n^2/(k^2-k)\} \\ &< Ae^{-An^2}. \end{aligned}$$

Next, by (1.3),

$$\begin{aligned} F_n - f_n &= \sum_{r=1}^{n-1} \binom{n-1}{r-1} F_{n-r} f_r \\ &\leqslant \sum_{r=1}^{n-1} \binom{n-1}{r-1} M_{n-r} M_r \\ &\leqslant \sum_{r=1}^{n-1} \binom{n-1}{r-1} k^n T\{K(n-r)^2 + Kr^2\} \end{aligned}$$

and, again assuming Theorem 1, we have

$$\frac{F_n - f_n}{M_n} \leqslant An^{\frac{1}{2}(k-1)} \sum_{r=1}^{n-1} \binom{n-1}{r-1} T\{-2Kr(n-r)\}.$$

Now

$$\begin{aligned} &\sum_{r=1}^{n-1} \binom{n-1}{r-1} T\{-2Kr(n-r)\} \\ &\quad \leqslant \sum_{r=1}^{[\frac{1}{2}n]} \binom{n-1}{r-1} T(-Krn) + \sum_{s=1}^{[\frac{1}{2}(n-1)]} \binom{n-1}{s} T(-Ksn) \\ &\quad < 2^{-Kn}(1 + 2^{-Kn})^{n-1} + (1 + 2^{-Kn})^{n-1} - 1 \\ &\quad = (1 + 2^{-Kn})^n - 1 < n2^{-Kn}(1 + 2^{-Kn})^{n-1} < An2^{-Kn}. \end{aligned}$$

Hence

$$0 < (F_n - f_n)/M_n < An2^{-Kn} < Ae^{-An}.$$

Theorem 2 follows from this and (3.4).

4. Proof of Theorem 1. Next we prove Theorem 1. We write

$$P = P(s_1, s_2, \ldots, s_k) = \frac{n!}{s_1! \ldots s_k!}.$$

By (1.1) and (3.3), we have

$$T(-Kn^2)M_n = \sum_{(n)} PT(-\tfrac{1}{2}k^{-2}S_2) = \sum{}' + \sum{}'', \tag{4.1}$$

where $\sum'$ includes all those terms for which $|ks_i - n| < n^{\frac{1}{2}}$ for every i. For every term in $\sum''$, we have $|ks_i - n| \geqslant n^{\frac{1}{2}}$ for at least one value of i and so $S_2 \geqslant n^{\frac{1}{2}}$. We have then

$$\begin{aligned} \sum{}'' &\leqslant T(-An^{\frac{1}{2}}) \sum{}'' P(s_1, \ldots, s_k) \\ &< T(-An^{\frac{1}{2}}) \sum_{(n)} P = k^n T(-An^{\frac{1}{2}}). \end{aligned} \tag{4.2}$$

Now we consider any term of $\sum'$, so that $s_i > A_n$ for every i. By Stirling's formula, for $n > A$,

$$\log(n!) = (n + \tfrac{1}{2}) \log n - n + \tfrac{1}{2} \log(2\pi) + \sum_{h=1}^{H-1} c_h n^{-h} + O(n^{-H})$$

and so

$$\begin{aligned} \log P = (n + \tfrac{1}{2}) \log n - \sum (s_i + \tfrac{1}{2}) \log s_i - \tfrac{1}{2}(k-1) \log(2\pi) \\ + \sum_{h=1}^{H-1} c_h (n^{-h} - \sum s_i^{-h}) + O(n^{-H}). \end{aligned}$$

Also

$$\begin{aligned} (n + \tfrac{1}{2}) \log n - \sum (s_i + \tfrac{1}{2}) \log s_i \\ &= (n + \tfrac{1}{2}) \log n - \sum (s_i + \tfrac{1}{2}) \log(n/k) - \sum_1 \\ &= -\tfrac{1}{2}(k-1) \log n + (n + \tfrac{1}{2}k) \log k - \sum_1, \end{aligned}$$

where

$$\begin{aligned} \sum_1 &= \frac{n}{k} \sum \left(1 + \frac{u_i + \frac{1}{2}k}{n}\right) \log\left(1 + \frac{u_i}{n}\right) \\ &= \sum_{m=2}^{\infty} \frac{(-1)^{m-2}}{n^{m-1}} \left\{ \frac{S_{m-1}}{2(m-1)} + \frac{S_m}{km(m-1)} \right\}, \end{aligned}$$

since $S_1 = 0$ by (3.1). Again

$$n^h(n^{-h} - \sum s_i^{-h}) = 1 - k^h \sum \{1 + (u_i/n)\}^{-h}$$
$$= 1 - k^h\left\{k + \sum_{m=1}^{\infty} (-1)^m \binom{m+h-1}{h-1} \frac{S_m}{n^m}\right\}$$

and so

$$n^{-h} - \sum s_i^{-h} = -\frac{k^{h+1} - 1}{n^h} - \sum_{m=1}^{\infty} (-1)^m \binom{m+h-1}{h-1} \frac{S_m}{n^{h+m}}.$$

Hence, if we take H odd,

$$\log P = (n + \tfrac{1}{2}k)\log k - \tfrac{1}{2}(k-1)\log(2\pi n) + \sum_{h=1}^{H-1} d_h n^{-h} + O\{n^{-H}(1 + S_{H+1})\}, \tag{4.3}$$

where

$$d_h = \sum_{m=0}^{h+1} v(k, h, m) S_m$$

is a polynomial in the u_i of degree at most $h+1$. Now

$$\exp\left(\sum_{h=1}^{H-1} d_h n^{-h}\right) = 1 + \sum_{h=1}^{\infty} D_h n^{-h},$$

where

$$D_h = \sum_{\sum m_c c = h} \prod_c \left(\frac{d_c^{m_c}}{m_c!}\right),$$

the sum being taken over all partitions of h, a typical partition being into m_1 parts 1, m_2 parts 2, and so on, and the product over every different part c in the partition. Thus D_h is a polynomial in the u_i of degree at most $2h$ and

$$D_h \leqslant A_h(1 + S_{2h}). \tag{4.4}$$

Hence, by (4.3),

$$\sum{}' PT\left(-\frac{S_2}{2k^2}\right) = \frac{k^{n+\frac{1}{2}k}}{(2\pi n)^{\frac{1}{2}(k-1)}}\left\{\sum_{h=0}^{H-1} \frac{J_h'}{n^h} + O\left(\frac{J_0' + V_{2H}'}{n^H}\right)\right\}, \tag{4.5}$$

where

$$V_m' = \sum{}' S_m T(-\tfrac{1}{2}k^{-2}S_2)$$

and

$$J_0' = V_0', \quad J_h' = \sum{}' D_h T(-\tfrac{1}{2}k^{-2}S_2).$$

We write

$$V_m = V_m(n) = \sum_{((n))} S_m T(-\tfrac{1}{2}k^{-2}S_2)$$

with the notation introduced in (2.3). The sum is certainly convergent and

$$(4.6)\qquad |V_m - V_m'| \leqslant \sum_{s_1 \geqslant n^{\frac{1}{4}}} |S_m| T(-\tfrac{1}{2}k^{-2}S_2)$$
$$< A_m \sum_{t \geqslant n^{\frac{1}{4}}} t^{m+\frac{1}{2}} T(-At) < A_m T(-An^{1/4}).$$

We see that, when $m > 0$,

$$S_m(s_1, s_2, \ldots, s_k; n) = \sum (ks_i - n)^m = \sum \{k(s_i - 1) - (n-k)\}^m$$
$$= S_m(s_1 - 1, s_2 - 1, \ldots, s_k - 1; n - k)$$

and so, when $m \geqslant 0$,

$$(4.7)\qquad V_m(n) = V_m(n-k),$$

from which it follows that

$$(4.8)\qquad |V_m(n)| \leqslant \max_{|a| \leqslant \frac{1}{2}k} |V_m(a)| < A.$$

We write

$$J_0 = V_0,\ J_k = J_k(n) = \sum_{((n))} D_k T(-\tfrac{1}{2}k^{-2}S_2),$$

the convergence of the last sum following from that of V_0 and V_{2k} by (4.4). Also

$$|J_k - J_k'| < A_k T(-An^{1/4})$$

by (4.4) and (4.6). Hence, by (4.1), (4.5), and (4.8),

$$(4.9)\qquad M_n = (2\pi n)^{-\frac{1}{2}(k-1)} k^{n+\frac{1}{2}k} T(Kn^2) \left\{ \sum_{h=0}^{H-1} J_h n^{-h} + O(n^{-H}) \right\}.$$

We can show, just as for $V_m(n)$ in the last paragraph that $J_k(n) = J_k(n-k)$ and so $|J_k(n)| < A_k$. (4.9) is now Theorem 1 with

$$C_k = C_k(k, n) = k^{\frac{1}{2}} (\log 2)^{\frac{1}{2}(k-1)} (2\pi)^{-\frac{1}{2}(k-1)} J_k$$

and $C_k(k, n) = C_k(k, a)$, if a is any residue of $n \pmod k$.

By (3.2) and (2.3),

$$J_0(n) = V_0(n) = L(n).$$

Hence, by (4.7), $L(n) = L(a)$ and (2.2) follows.

5. Proof of Theorem 3. We now evaluate $C_0(k, a)$ and $L(a)$, which are defined by (2.2) and (2.3). We suppose (2.4) satisfied and write

$$T_r = a - \sum_{i=r+1}^{k-1} s_i \qquad (0 \leqslant r \leqslant k-2), \qquad T_{k-1} = a.$$

We can prove by induction on r that, for $1 \leqslant r \leqslant k-1$, we have

$$(5.1)\qquad s_k^2 + \sum_{i=1}^{r} s_i^2 = \sum_{i=1}^{r} \frac{\{(i+1)s_i - T_i\}^2}{i(i+1)} + \frac{T_r^2}{r+1}.$$

For $r = 1$, (5.1) reduces to

$$s_k^2 + s_1^2 = \tfrac{1}{2}(2s_1 - T_1)^2 + \tfrac{1}{2}T_1^2,$$

which is true since

$$T_1 = a - \sum_{i=2}^{k-1} s_i = s_1 + s_k.$$

If we assume (5.1) true for $r = R - 1$, its truth for $r = R$ follows provided that

$$R(R+1)s_R^2 = \{(R+1)s_R - T_R\}^2 + RT_R^2 - (R+1)T_{R-1}^2$$

and this is a trivial consequence of the fact that $T_R = T_{R-1} + s_R$.

If we put $r = k - 1$ in (5.1), we have

$$\sum s_i^2 = \sum_{i=1}^{k-1} 2A_i(s_i + y_i)^2 + (a^2/k),$$

where

$$A_i = \frac{i+1}{2i}, \qquad y_i = -\frac{T_i}{i+1} = \frac{1}{i+1}\left(\sum_{r=i+1}^{k-1} s_r - a\right).$$

Hence

$$L(a) = \sum_{((a))} T\{-\tfrac{1}{2}\sum s_i^2 + \tfrac{1}{2}(a^2/k)\} \tag{5.2}$$

$$= \sum_{s_1,\ldots,s_{k-1}=-\infty}^{\infty} T\left\{-\sum_{i=1}^{k-1} A_i(s_i + y_i)^2\right\}.$$

We have thus eliminated s_k.

We now take $x > 0$ and write

$$W(x, y) = \sum_{s=-\infty}^{\infty} e^{-\pi(s+y)^2/x}, \qquad W(x) = W(x, 0).$$

An application of Poisson's formula **(2)** gives us

$$W(x, y) = \left(\frac{\pi}{x}\right)^{\frac{1}{2}}\left\{1 + 2\sum_{t=1}^{\infty} \exp\left(-\frac{\pi^2 t^2}{x}\right)\cos 2\pi t y\right\}. \tag{5.3}$$

If we put $y = 0$ in this, we have

$$W(x) = (\pi/x)^{\frac{1}{2}} W(\pi^2/x).$$

It follows that

$$\left|\left(\frac{x}{\pi}\right)^{\frac{1}{2}} W(x, y) - 1\right| \leqslant 2\sum_{t=1}^{\infty} \exp\left(-\frac{\pi^2 t^2}{x}\right)|\cos 2\pi t y| \tag{5.4}$$

$$\leqslant 2\sum_{t=1}^{\infty} \exp\left(-\frac{\pi^2 t^2}{x}\right) = W\left(\frac{\pi^2}{x}\right) - 1.$$

We now write

$$x_i = A_i \log 2, \quad B_i = W(\pi^2/x_i) - 1,$$

so that, by (5.4), we have

$$1 - B_t < (x_t/\pi)^{\frac{1}{2}} W(x_t, y_t) < 1 + B_t.$$

We have then, by (5.2),

$$L(a) = \sum_{s_2, s_3, \ldots, s_{k-1} = -\infty}^{\infty} W(x_1, y_1) T\left\{ - \sum_{t=2}^{k-1} A_t (s_t + y_t)^2 \right\}$$
$$< \left(\frac{\pi}{x_1}\right)^{\frac{1}{2}} (1 + B_1) \sum_{s_1, \ldots, s_{k-1} = -\infty}^{\infty} T\left\{ - \sum_{t=2}^{k-1} A_t (s_t + y_t)^2 \right\},$$

since B_1 is independent of y_1 and so of $s_2, \ldots, s_{k-1}$ and all the terms in the last sum are positive. Continuing this process step by step, we find that

$$L(a) < \pi^{\frac{1}{2}(k-1)} \prod_{t=1}^{k-1} \{x_t^{-\frac{1}{2}} (1 + B_t)\}.$$

Now

$$\prod_{t=1}^{k-1} x_t = (\log 2)^{k-1} \prod_{t=1}^{k-1} A_t = \frac{k (\log 2)^{k-1}}{2^{k-1}}$$

and so

$$C_0(k, a) = k^{\frac{1}{2}} \left(\frac{\log 2}{2\pi}\right)^{\frac{1}{2}(k-1)} L(a) < \prod_{t=1}^{k-1} (1 + B_t).$$

A precisely similar argument, with inequality signs reversed, shows that

$$C_0(k, a) > \prod_{t=1}^{k-1} (1 - B_t).$$

The B_t are very easy to compute. We find that

$$B_1 = 1.3097 \times 10^{-6}, \qquad B_2 = 1.1374 \times 10^{-8}$$

and so on; in particular,

$$B_{200} < 10^{-12}.$$

Theorem 3 follows quite simply from the calculations.

On the other hand, if $k = 2$ and $s_1 + s_2 = a$,

$$s_1^2 + s_2^2 - \tfrac{1}{2}a^2 = \tfrac{1}{2}(s_1 - s_2)^2 = \tfrac{1}{2}(2s_1 - a)^2$$

and so

$$L(a) = \sum_{s=-\infty}^{\infty} 2^{-(s - \frac{1}{2}a)^2} = W(\log 2, -\tfrac{1}{2}a).$$

Hence

$$C_0(2, a) = \left(\frac{\log 2}{\pi}\right)^{\frac{1}{2}} W(\log 2, -\tfrac{1}{2}a) = 1 + 2 \sum_{m=1}^{\infty} \exp\left(-\frac{m^2 \pi^2}{\log 2}\right) \cos m\pi a$$

by (5.3) and so

$$C_0(2, 0) - C_0(2, 1) = 4\sum_{m=0}^{\infty} \exp\left(-\frac{(2m+1)^2\pi^2}{\log 2}\right)$$

$$> 4\exp(-\pi^2/\log 2) > 2.6194 \times 10^{-6}.$$

(2.5) follows and we observe also that $C_0(2, 0)$ and $C_0(2, 1)$ differ by very nearly as much as Theorem 3 allows.

References

1. R. C. Read, *The number of k-coloured graphs on labelled nodes*, Can. J. Math., *12* (1960), 410–414.
2. E. C. Titchmarsh, *Fourier Integrals* (Oxford, 1937), 64.

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A NORMAL FORM FOR A MATRIX UNDER THE UNITARY CONGRUENCE GROUP

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1. Introduction. Let C be a square matrix with complex elements. If $C = C'$ (C' denotes the transpose of C) there exists a unitary matrix U such that

$$U'CU = \operatorname{diag}[\mu_1, \mu_2, \ldots, \mu_n] \tag{1}$$

where the μ's are the non-negative square roots of the eigenvalues $\mu_1^2, \mu_2^2, \ldots, \mu_n^2$ of C^*C (C^* is the adjoint of C) **(2)**. If C is skew-symmetric, that is, $C = -C'$, there exists a unitary matrix U such that

$$U'CU = \Sigma_1 \dotplus \Sigma_2 \dotplus \ldots \dotplus \Sigma_k \dotplus 0 \dotplus \ldots \dotplus 0 \tag{2}$$

where

$$\Sigma_r = \begin{bmatrix} 0 & \alpha_r \\ -\alpha_r & 0 \end{bmatrix}, \quad \alpha_r > 0, \quad r = 1, 2, \ldots, k, \tag{3}$$

and the α's are the positive square roots of the non-zero eigenvalues $\alpha_1^2, \alpha_2^2, \ldots, \alpha_k^2$ of C^*C **(1)**. Clearly rank $C = 2k$ and the number of zeros appearing in (2) is $n - 2k$. Both (1) and (2) are classical. In a recent paper **(3)** Stander and Wiegman, apparently unaware of **(1)**, give an alternative derivation of (2) with its appropriate generalization to quaternions.

The forms appearing on the right-hand sides of (1) and (2) possess the distinction of actually being canonic for symmetric and skew-symmetric matrices, respectively. The problem of finding a canonic form for an arbitrary matrix under the group of unitary congruence transformations is not only of mathematical interest but of the utmost importance for applied electrical engineering network theory, representing as it does the physical operation of embedding an n-port in a lossless, reciprocal "all-pass" $2n$-port **(4; 5)**. In this paper a start is made on this problem by obtaining a normal form for an arbitrary matrix under a U', U transformation and it is shown that (1) and (2) are immediate corollaries. Now $U'CU = W$, U unitary, implies that $U^*\bar{C}CU = \bar{W}W = U^{-1}\bar{C}CU$. Thus $\bar{C}C$ undergoes a similarity transformation and it is not surprising that a leading role is played by the eigenvalues of $\bar{C}C$. It is also shown that the normal form of C collapses into a direct sum of matrices if $\bar{C}C$ is normal. A set of sufficient conditions is given for this form to be canonic.

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The technique employed is quite elementary and is inspired by an idea used by Schur in his treatment of the symmetric case **(2)**.

2. Notation and preliminary results. Let A be an arbitrary matrix. Then A', $\bar{A}$, A^*, A^{-1}, $d(A)$, and $r(A)$ denote the transpose, the complex conjugate, the adjoint $\bar{A}'$, the inverse, the determinant, and the rank of A respectively. Column vectors are indicated by $\mathbf{a}$, $\mathbf{b}$ etc., or in the alternative fashion $\mathbf{a} = (a_1, a_2, \ldots, a_n)'$ whenever it is desirable to exhibit the components explicitly. $\mathbf{0}_n$, $0_{m,n}$, and 1_n represent, in the same order, the n-dimensional zero vector, the $m \times n$ zero matrix, and $n \times n$ identity matrix.

Consider a general $n \times n$ matrix of complex elements $C = B + iD$ where B and D are its real and imaginary parts. To C can be assigned the real $2n \times 2n$ matrix

$$T(C) = \left[\begin{array}{c|c} B & D \\ \hline D & -B \end{array}\right]. \tag{4}$$

The properties of $T(C)$ are subsumed in Lemma 1 **(2)**.

LEMMA 1. (1) *The eigenvalues of $T(C)$ occur in negative pairs.* (2) *The eigenvalues of $T(C)$ occur in complex conjugate pairs.* (3) *Let the matrix $\bar{C}C$ possess the distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$ with respective multiplicities $r_1, r_2, \ldots, r_r$. Then the distinct eigenvalues of $T^2(C)$ are $\lambda_1, \lambda_2, \ldots, \lambda_r$ with respective multiplicities $2r_1, 2r_2, \ldots, 2r_r$.*

Proof. By direct verification

$$\sigma_1 T(C) \sigma_1^{-1} = -T(C) \tag{5}$$

where

$$\sigma_1 = \left[\begin{array}{c|c} 0 & 1_n \\ \hline -1_n & 0 \end{array}\right], \qquad \sigma_1^2 = -1_{2n}. \tag{6}$$

(2) Since $T(C)$ is real, its characteristic equation has real coefficients. (3) A straightforward calculation yields

$$\sigma_2 T^2(C) \sigma_2^{-1} = \bar{C}C + C\bar{C}$$

where

$$\sigma_2 = \frac{1}{\sqrt{2}}\left[\begin{array}{c|c} 1_n & -i1_n \\ \hline -i1_n & 1_n \end{array}\right], \qquad \sigma_2 \sigma_2^* = 1_{2n},$$

Q.E.D.

LEMMA 2. *Let $C = B + iD$ (B and D real) be an $n \times n$ matrix. There exists a non-trivial n-vector $\mathbf{a}$ and a scalar ξ such that*

$$C\mathbf{a} = \xi\bar{\mathbf{a}}$$

if and only if $\bar{C}C$ possesses a non-negative eigenvalue.

Proof. Suppose (9) is satisfied for some non-trivial $\mathbf{a}$. Then,

$$\bar{C}C\mathbf{a} = \xi\bar{C}\bar{\mathbf{a}} = |\xi|^2\mathbf{a},$$

that is, $\mathbf{a}$ is an eigenvector of $\bar{C}C$ corresponding to the non-negative eigenvalue $|\xi|^2$. Conversely if $\bar{C}C$ has an eigenvalue $\xi^2 > 0$, $\xi > 0$, it follows, by Lemma 1, parts (1) and (3), that $\pm\xi$ are both eigenvalues of $T(C)$. Thus there exists a real $2n$-vector

$$\psi = \begin{bmatrix} \mathbf{a}_1 \\ \hline -\mathbf{a}_2 \end{bmatrix} n \tag{7}$$

such that

$$\left[\begin{array}{c|c} B & D \\ \hline D & -B \end{array}\right]\psi = \xi\psi, \tag{8}$$

or, in expanded form

$$\begin{aligned} \xi\mathbf{a}_1 &= B\mathbf{a}_1 - D\mathbf{a}_2, \\ -\xi\mathbf{a}_2 &= D\mathbf{a}_1 + B\mathbf{a}_2. \\ \therefore \xi(\mathbf{a}_1 - i\mathbf{a}_2) &= (B + iD)(\mathbf{a}_1 + i\mathbf{a}_2). \\ \therefore \quad \xi\bar{\mathbf{a}} &= C\mathbf{a}, \qquad (9) \\ \mathbf{0} \neq \mathbf{a} &= \mathbf{a}_1 + i\mathbf{a}_2, \qquad (10) \end{aligned}$$

Q.E.D.

LEMMA 3. *Let $\bar{C}C$ have the non-negative eigenvalues $\xi_1^2, \xi_2^2, \ldots, \xi_\nu^2$, $\xi_r > 0$, $r = 1, 2, \ldots, \nu$. Then there exists a unitary matrix U such that*

$$U'CU = \begin{matrix} \nu & n-\nu \end{matrix} \left[\begin{array}{c|c} \Delta & C_1 \\ \hline O & C_2 \end{array}\right] \begin{matrix} \nu \\ n-\nu \end{matrix} \tag{11}$$

where Δ is a $\nu \times \nu$ upper triangular matrix with diagonal elements $\xi_1, \xi_2, \ldots, \xi_\nu$.

Proof. According to Lemma 2 there exists a non-zero n-vector $\mathbf{a}_1$ such that $\mathbf{a}_1^*\mathbf{a}_1 = 1$ and $C\mathbf{a}_1 = \xi_1\bar{\mathbf{a}}_1$. Let A be an $n \times n$ unitary matrix with columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$. Then,

$$\begin{aligned} (A'CA)_{k,1} &= \mathbf{a}_k'C\mathbf{a}_1 = \xi_1\mathbf{a}_k'\bar{\mathbf{a}}_1 \\ &= 1, \qquad k = 1, \\ &= 0, \qquad k > 1. \end{aligned}$$

Hence

$$A'CA = \begin{matrix} 1 & n-1 \end{matrix} \left[\begin{array}{c|c} \xi_1 & M \\ \hline O & N \end{array}\right] \begin{matrix} 1 \\ n-1 \end{matrix} \tag{12}$$

and

$$A^*\bar{C}CA = \begin{matrix} 1 & n-1 \end{matrix} \left[\begin{array}{c|c} \xi_1^2 & P \\ \hline O & \bar{N}N \end{array}\right] \begin{matrix} 1 \\ n-1 \end{matrix}, \tag{13}$$

where

$$P = \xi_1 M + \bar{M} N. \tag{14}$$

Now N is an $(n-1) \times (n-1)$ matrix and since the remaining eigenvalues of $\bar{C}C$ and $\bar{N}N$ coincide, the proof of the lemma is very easily completed by induction. Let V be an $(n-1) \times (n-1)$ unitary matrix such that

$$V'NV = \begin{array}{c} \begin{matrix} \nu - 1 & n - \nu \end{matrix} \\ \left[\begin{array}{c|c} \delta & \hat{M} \\ \hline 0 & C_2 \end{array}\right] \begin{matrix} \nu - 1 \\ n - \nu \end{matrix} \end{array}$$

where δ is a $(\nu - 1) \times (\nu - 1)$ upper triangular matrix with the diagonal entries $\xi_2, \xi_3, \ldots, \xi_\nu$. Then, the $n \times n$ matrix

$$U = A\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & V \end{array}\right]$$

is unitary and

$$U'CU = \begin{array}{c} \left[\begin{array}{c|c|c} \xi_1 & E & F \\ \hline 0 & \delta & \hat{M} \\ \hline 0 & 0 & C_2 \end{array}\right] \begin{matrix} 1 \\ \nu - 1 \\ n - \nu \end{matrix} \end{array}$$

$$= \left[\begin{array}{c|c} \Delta & C_1 \\ \hline 0 & C_2 \end{array}\right]$$

in which

$$MV = [E|F],$$

$$\Delta = \begin{array}{c} \left[\begin{array}{c|c} \xi_1 & E \\ \hline 0 & \delta \end{array}\right] \begin{matrix} 1 \\ \nu - 1 \end{matrix} \end{array}$$

and

$$C_1 = \left[\begin{array}{c} F \\ \hline \hat{M} \end{array}\right],$$

Q.E.D.

By applying Lemma 3 to C' it is easily established that there is also a unitary matrix V such that

$$V'CV = \begin{array}{c} \left[\begin{array}{c|c} \hat{\Delta} & O \\ \hline Q & R \end{array}\right] \begin{matrix} \nu \\ n - \nu \end{matrix}, \\ \begin{matrix} \nu & n - \nu \end{matrix} \end{array} \tag{15}$$

$\hat{\Delta}$ being a lower triangular $\nu \times \nu$ matrix whose diagonal elements are $\xi_1, \xi_2, \ldots, \xi_\nu$.

LEMMA 4. *Let* $-\lambda^2 \geq 0$ *be an eigenvalue of* $\bar{C}C$. *There exist two linearly independent n-vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *such that*

(16) $$C\mathbf{a} = \lambda\bar{\mathbf{b}}$$

and

(17) $$C\mathbf{b} = -\lambda\bar{\mathbf{a}}.$$

Proof. By hypothesis $-\lambda^2 \geq 0$ is an eigenvalue of $\bar{C}C$. Let $\mathbf{a}$ be an associated eigenvector:

(18) $$\bar{C}C\mathbf{a} = -\lambda^2\mathbf{a}.$$

Define the vector $\mathbf{b}$ by

(19) $$C\mathbf{a} = \lambda\bar{\mathbf{b}}.$$

From (19) and (18)

$$\bar{C}C\mathbf{a} = \lambda\bar{C}\bar{\mathbf{b}} = -\lambda^2\mathbf{a}\,.$$
$$\therefore C\mathbf{b} = -\bar{\lambda}\bar{\mathbf{a}}.$$

Now suppose that $\mathbf{a}$ and $\mathbf{b}$ are linearly dependent, that is, $\mathbf{b} = \mu\mathbf{a}$, μ a scalar. Substituting in (19), $C\mathbf{a} = \lambda\bar{\mu}\bar{\mathbf{a}}$, whence, using (19) once again,

(20) $$\bar{C}C\mathbf{a} = \lambda\bar{\mu}\bar{C}\bar{\mathbf{a}} = |\lambda\mu|^2\mathbf{a}.$$

Equation (20) shows that $\mathbf{a}$ is also an eigenvector of $\bar{C}C$ corresponding to the positive eigenvalue $|\lambda\mu|^2 \neq -\lambda^2$. But this is impossible because eigenvectors belonging to distinct characteristic values must be linearly independent, Q.E.D.

Since $\bar{C}C\mathbf{b} = -(\bar{\lambda})^2\mathbf{b}$, $\mathbf{b}$ is also an eigenvector of $\bar{C}C$ with eigenvalue $-(\bar{\lambda})^2$. If λ is real $\mathbf{a}$ and $\mathbf{b}$ are two linearly independent eigenvectors of $\bar{C}C$ corresponding to the same negative eigenvalue $-\lambda^2 < 0$. Hence any negative eigenvalue of a matrix of the form $\bar{C}C$ is at least of algebraic multiplicity two. It will soon appear that its multiplicity is always an even integer. More important is the observation that for $-\lambda^2 < 0$, $\lambda > 0$, it may always be assumed, without loss of generality, that $\mathbf{a}^*\mathbf{b} = 0$. For, if $C\mathbf{a}_0 = \lambda\bar{\mathbf{b}}_0$ and $C\mathbf{b}_0 = -\lambda\bar{\mathbf{a}}_0$, $C\mathbf{x} = \lambda\bar{\mathbf{y}}$, where $\mathbf{x} = \mathbf{a}_0 + \mu\mathbf{b}_0$, $\mathbf{y} = \mathbf{b}_0 - \bar{\mu}\mathbf{a}_0$ and μ is an arbitrary scalar. By choosing μ to be one of the roots of the quadratic equation

$$(\mathbf{a}_0^*\mathbf{b}_0)\mu^2 + (\mathbf{a}_0^*\mathbf{a}_0 - \mathbf{b}_0^*\mathbf{b}_0)\mu - \overline{\mathbf{a}_0^*\mathbf{b}_0} = 0,$$

$\mathbf{x}$ can be made orthogonal to $\mathbf{y}$. Since $\mathbf{x}$ is also an eigenvector of $\bar{C}C$ associated with the eigenvalue $-\lambda^2 < 0$, the result follows.

If λ is complex and $-\lambda^2 \geq 0$, one resorts to the Schmidt process to create an orthonormal pair $\mathbf{x}$ and $\mathbf{y}$ (choose $\mathbf{a}^*\mathbf{a} = 1$ from the outset). Let

(21) $$\mathbf{a} = \mathbf{x}$$

and

(22) $$\mathbf{b} = \mu\mathbf{x} + \eta\mathbf{y},$$

μ and η being scalars. The conditions $\mathbf{x}^*\mathbf{y} = 0$, $\mathbf{x}^*\mathbf{x} = 1$ yield

(23) $$\mu = \mathbf{a}^*\mathbf{b},$$

(24) $$|\eta|^2 = (\mathbf{b} - \mu\mathbf{a})^*(\mathbf{b} - \mu\mathbf{a}) > 0.$$

Equation (24) defines only the magnitude of η which may therefore be chosen real and positive. Similarly, by changing the argument of $\mathbf{a}$ suitably, $\mu \geqslant 0$. As has already been pointed out, if $\lambda > 0$, $\mu = 0$. Substitution of (21) and (22) into (16) and (17) gives

$$\mathbf{x}'C\mathbf{x} = \lambda\mu, \tag{25}$$

$$\mathbf{y}'C\mathbf{x} = \lambda\eta, \tag{26}$$

$$\mathbf{x}'C\mathbf{y} = -\frac{1}{\eta}(\bar{\lambda} + \lambda\mu^2), \tag{27}$$

$$\mathbf{y}'C\mathbf{y} = -\lambda\mu, \tag{28}$$

$\eta > 0$, $\mu \geqslant 0$.

Let A be an $n \times n$ unitary matrix with $\mathbf{x}$ and $\mathbf{y}$ for its first two columns. Then, by equations (25)–(28),

$$A'CA = \begin{array}{c} \begin{matrix} 2 & n-2 \end{matrix} \\ \left[\begin{array}{c|c} \Sigma & C_1 \\ \hline O & C_2 \end{array}\right] \begin{matrix} 2 \\ n-2 \end{matrix} \end{array} \tag{29}$$

where

$$\Sigma = \left[\begin{array}{c|c} \lambda\mu & -\frac{1}{\eta}(\bar{\lambda} + \lambda\mu^2) \\ \hline \lambda\eta & -\lambda\mu \end{array}\right], \qquad \mu \geqslant 0,\ \eta > 0. \tag{30}$$

If $\lambda > 0$, $\mu = 0$ and

$$\Sigma = \left[\begin{array}{c|c} O & -\frac{\lambda}{\eta} \\ \hline \lambda\eta & O \end{array}\right], \qquad \lambda > 0,\ \eta > 0. \tag{31}$$

Observe that

$$A^*\bar{C}CA = \left[\begin{array}{c|c} \overline{\Sigma}\,\Sigma & \overline{\Sigma}\,C_1 + \bar{C}_1C_2 \\ \hline O & \bar{C}_2C_2 \end{array}\right] \tag{31a}$$

and

$$\overline{\Sigma}\,\Sigma = \left[\begin{array}{c|c} -\lambda^2 & \frac{\mu}{\eta}(\lambda^2 - \bar{\lambda}^2) \\ \hline O & -(\bar{\lambda})^2 \end{array}\right]. \tag{31b}$$

A repetition of the process on C_2, etc., leads to the conclusion that any negative eigenvalue of $\bar{C}C$ is of even multiplicity because each successive step contributes two identical ones. Combining this with the fact that the eigenvalues of $\bar{C}C$ occur in complex conjugate pairs leads to Lemma 5.

LEMMA 5. *Let C be an arbitrary $n \times n$ matrix. Those eigenvalues $-\lambda^2$ of $\bar{C}C$ which do not lie in the half-line $-\lambda^2 \geqslant 0$ may be arranged in the tableau*

$$\begin{array}{l} -\lambda_1^2,\ -\lambda_2^2, \ldots, -\lambda_k^2, \\ -(\bar{\lambda}_1)^2,\ -(\bar{\lambda}_2)^2, \ldots, -(\bar{\lambda}_k)^2 \end{array} \tag{32}$$

and there exists a unitary matrix U such that

$$U'CU = \left[\begin{array}{c|c} \Omega & C_1 \\ \hline O & C_2 \end{array}\right] \begin{matrix} 2k \\ n-2k \end{matrix} \tag{33}$$
$$\qquad\quad 2k \quad n-2k$$

where

$$\Omega = \begin{bmatrix} \Sigma_1 & x & x & x \\ 0 & \Sigma_2 & x & x \\ 0 & 0 & \ddots & x \\ 0 & 0 & 0 & \Sigma_k \end{bmatrix} \begin{matrix} \\ \\ \\ 2k \end{matrix} \tag{34}$$
$$\qquad 2k$$

is a block upper triangular matrix in which the 2×2 Σ's are defined in (30) *and the crosses indicate the possible presence of non-zero 2×2 blocks. Observe that* (31b) *implies that $\bar{\Omega}\Omega$ is upper triangular.*

Proof. Induction.

Enough material is now on hand for the main theorem.

3. A normal form for a matrix under the unitary congruence group.

THEOREM 1. *Let C be an arbitrary $n \times n$ matrix. The eigenvalues of $\bar{C}C$ may be arranged as follows:*

$$\begin{array}{ll} \xi_1^2, \xi_2^2, \ldots, \xi_\nu^2, \xi_r > 0, & r = 1, 2, \ldots, \nu, \\ -\lambda_1^2, -\lambda_2^2, \ldots, -\lambda_k^2, -\lambda_j^2 \geq 0, & j = 1, 2, \ldots, k, \\ -(\bar{\lambda}_1)^2, -(\bar{\lambda}_2)^2, \ldots, -(\bar{\lambda}_k)^2, & \nu + 2k = n. \end{array} \tag{35}$$

Furthermore, there exists an $n \times n$ unitary matrix U such that

$$U'CU = \left[\begin{array}{c|c} \Delta & C_1 \\ \hline 0 & \Omega \end{array}\right] \begin{matrix} \nu \\ 2k \end{matrix} \tag{36}$$
$$\qquad\quad \nu \quad 2k$$

where Δ is a $\nu \times \nu$ upper triangular matrix with diagonal elements $\xi_1, \xi_2, \ldots, \xi_\nu$ and Ω is a $2k \times 2k$ block upper triangular matrix possessing the structure (34) *in which the 2×2 Σ's are as defined in* (30) *for complex λ and* (31) *for $\lambda > 0$.*

Proof. Lemmas 3 and 5.

COROLLARY 1. *Let $C = C'$ be a symmetric $n \times n$ matrix. There exists an $n \times n$ unitary matrix U which reduces C to the diagonal form* (1).

Proof. Since $\bar{C}C = C^*C$, all eigenvalues of $\bar{C}C$ are non-negative. Hence $\nu = n$ and $k = 0$. By Theorem 1 there exists a unitary matrix U such that $U'CU = \Delta$, an $n \times n$ upper triangular matrix. But Δ must be symmetric and therefore is actually diagonal, Q.E.D.

COROLLARY 2. *Let $C = -C'$ be an arbitrary $n \times n$ skew-symmetric matrix. There exists an $n \times n$ unitary matrix U which reduces C to the direct sum form* (2).

Proof. Since $\bar{C}C = -C^*C$, all eigenvalues of $\bar{C}C$ are either negative or zero. Let U be that unitary matrix which transforms C into the right-hand side of (36). The skew-symmetry of C implies that $\Delta = 0_{r,r}$, $C_1 = 0_{r,2k}$, and Ω is the direct sum

$$\Omega = \Sigma_1 \dotplus \Sigma_2 \dotplus \ldots \dotplus \Sigma_k,$$

each Σ having the structure (31). Now the Σ's themselves must also be skew-symmetric and therefore all η's = 1. That is to say,

$$\Sigma_r = \left[\begin{array}{c|c} 0 & -\lambda_r \\ \hline \lambda_r & 0 \end{array}\right], \lambda_r > 0, \qquad r = 1, 2, \ldots, k,$$

Q.E.D.

COROLLARY 3. *Let C be an $n \times n$ matrix. There exists an $n \times n$ unitary matrix U such that $U'CU$ is either upper or lower triangular if and only if the eigenvalues of $\bar{C}C$ are all non-negative.*

COROLLARY 4. *Let C be an $n \times n$ matrix. There exists an $n \times n$ unitary matrix U such that $U'CU$ is a block upper triangular matrix Ω (see* (34)*), the Σ's being defined by* (31)*, if and only if all eigenvalues of $\bar{C}C$ are negative.*

In either of the two cases C symmetric or C skew-symmetric, $\bar{C}C$ is normal and it is reasonable to expect that (36) simplifies considerably even under this more general restriction. A complete answer is available and easily obtained with the aid of Lemma 6.

LEMMA 6. *The matrix*

$$Q = \left[\begin{array}{c|c} A & B \\ \hline O & C \end{array}\right]^{k}_{r} \tag{37}$$

$$k \quad r$$

is normal if and only if $B = O_{k,r}$ and A and C are normal.

Proof. The condition $QQ^* = Q^*Q$ yields the three matrix equations

$$A^*A - AA^* = BB^*, \tag{38}$$

$$A^*B = BC^*, \tag{39}$$

$$CC^* - C^*C = B^*B. \tag{40}$$

By (38), $\operatorname{trace}(BB^*) = 0$. Thus $B = O_{k,r}$, $A^*A = AA^*$, and $C^*C = CC^*$, Q.E.D.

One corollary of this lemma is that any normal block-upper or lower triangular matrix Ω (the Σ's can now be arbitrary square blocks) is a direct sum $\Omega = \Sigma_1 \dot{+} \Sigma_2 \dot{+} \ldots \dot{+} \Sigma_k$.

THEOREM 2. *Let C be an $n \times n$ matrix. There exists an $n \times n$ unitary matrix U such that*

$$U'CU = \Delta \dot{+} \Omega \tag{41}$$

where

(1) *Δ is square and upper triangular,*

(2) *$\bar{\Delta}\Delta$ is diagonal,*

(3) *Ω is a block upper triangular matrix of the form* (34), *the blocks Σ being defined generically by*

$$\Sigma = \left[\begin{array}{c|c} 0 & \frac{\bar{\lambda}}{\eta} \\ \hline \eta\lambda & 0 \end{array}\right], \quad \eta > 0, \tag{41a}$$

and

(4) *$\bar{\Omega}\Omega$ is diagonal,*

if and only if $\bar{C}C$ is normal.

Proof. Suppose such a U exists, Δ and Ω possessing the properties enumerated in (1) to (4). Then,

$$U^*\bar{C}CU = \bar{\Delta}\Delta \dot{+} \bar{\Omega}\Omega,$$

a diagonal matrix. Since $\bar{C}C$ is unitarily similar to a diagonal matrix it is normal.

Conversely, let $\bar{C}C$ be normal. By Theorem 1 there exists an $n \times n$ unitary matrix U such that

$$U'CU = \begin{array}{c} \\ \nu \end{array}\!\!\left[\begin{array}{c|c} \Delta & C_1 \\ \hline O & \Omega \end{array}\right]\!\!\begin{array}{c} \nu \\ 2k \end{array} \tag{42}$$

$$\qquad\qquad\quad\ \ 2k$$

where Δ is upper triangular and Ω is as described in (31) and (34). Now

$$U^*\bar{C}CU = \begin{array}{c} \\ \nu \end{array}\!\!\left[\begin{array}{c|c} \bar{\Delta}\Delta & \Delta C_1 + \bar{C}_1\Omega \\ \hline O & \bar{\Omega}\Omega \end{array}\right]\!\!\begin{array}{c} \nu \\ 2k \end{array} \tag{43}$$

$$\qquad\qquad\quad\ \ 2k$$

The normality of $\bar{C}C$ implies that of the right-hand side of (43). By Lemma 6,

$$\begin{array}{ll} & \text{(a)}\ \ \Delta C_1 + \bar{C}_1\Omega = O_{\nu,2k} \\ (44) & \text{(b)}\ \ \bar{\Delta}\Delta \text{ is normal;} \\ & \text{(c)}\ \ \bar{\Omega}\Omega \text{ is normal.} \end{array}$$

But $\bar{\Delta}\Delta$ and $\bar{\Omega}\Omega$ are already upper triangular, and must therefore be diagonal. According to (44),

$$\Delta C_1 = -\bar{C}_1\Omega$$

or

$$\bar{\Delta}\Delta C_1 = -\bar{\Delta}\bar{C}_1\Omega = C_1\bar{\Omega}\Omega. \tag{45}$$

However, the eigenvalues of $\bar{\Delta}\Delta$ are all non-negative and those of $\bar{\Omega}\Omega$ are either negative or complex. Thus $\bar{\Delta}\Delta$ and $\bar{\Omega}\Omega$ have no eigenvalues in common and by a well-known theorem the only solution of (44) is $C_1 = O_{\nu,2k}$. As regards (41a), refer to equations (23) and (24). Since $\bar{C}C$ is normal any two eigenvectors **a** and **b** corresponding to the complex characteristic values $-\lambda^2$ and $(-\bar{\lambda})^2$ must be orthogonal and therefore all μ's are zero and (30) collapses into (41a), Q.E.D.

COROLLARY. *A sufficient set of conditions for* (41) *to be canonic* (*under a prescribed ordering*) *is the following: any negative eigenvalue of* $\bar{C}C$ *is of algebraic multiplicity not exceeding two and all other eigenvalues are simple.*

Proof. Let $U'CU = \Delta \dotplus \Omega$ where (see Theorem 2 and (31b))

$$\bar{\Delta}\Delta = \operatorname{diag}[\xi_1^2, \xi_2^2, \ldots, \xi_\nu^2] = D_1, \quad \xi_i > 0, \qquad (i = 1, 2, \ldots, \nu), \tag{46}$$

and

$$\bar{\Omega}\Omega = -\operatorname{diag}[\lambda_1^2, (\bar{\lambda}_1)^2, \lambda_2^2, (\bar{\lambda}_2)^2, \ldots, \lambda_k^2, (\bar{\lambda}_k)^2] = -D_2. \tag{47}$$

From (46), $\Delta\bar{\Delta}\Delta = \Delta D_1 = \bar{D}_1\Delta = D_1\Delta$. Thus,

$$(\Delta)_{rj}\xi_j^2 = (\Delta)_{rj}\xi_r^2, \qquad (r, j = 1, 2, \ldots, \nu). \tag{48}$$

Since, by hypothesis, $\xi_r^2 \neq \xi_j^2$, $r \neq j$, $(\Delta)_{rj} = 0$, $r \neq j$ and $\Delta = \operatorname{diag}[\xi_1, \xi_2, \ldots, \xi_\nu]$.

Again, from (47), $-\Omega\bar{\Omega}\Omega = \Omega D_2 = \bar{D}_2\Omega$. Now Ω is of even order $2k$ and may be partitioned into 2×2 blocks. For example, if $2k = 6$

$$\Omega = \begin{matrix} \\ 2 \\ 2 \\ 2 \end{matrix}\!\!\begin{bmatrix} \overset{2}{\Sigma_1} & \overset{2}{P} & \overset{2}{Q} \\ O & \Sigma_2 & R \\ O & O & \Sigma_3 \end{bmatrix}. \tag{49}$$

The matrix equation $\Omega D_2 = \bar{D}_2\Omega$ gives

$$P \operatorname{diag}[\lambda_2^2, (\bar{\lambda}_2)^2] = \operatorname{diag}[(\bar{\lambda}_1)^2, \lambda_1^2]P.$$

From the assumptions, $\bar{\lambda}_2^2 \neq \lambda_1^2 \neq \lambda_2^2$ and therefore $P = O_{2,2}$. All other 2×2 blocks are treated in the same way and shown to be zero matrices. Consequently

$$U'CU = \xi_1 \dotplus \xi_2 \dotplus \ldots \dotplus \xi_\nu \dotplus \Sigma_1 \dotplus \Sigma_2 \dotplus \ldots \dotplus \Sigma_k \tag{50}$$

where

$$\Sigma_r = \left[\begin{array}{c|c} O & \bar{\lambda}_r \\ \hline -\eta_r\lambda_r & \eta_r \\ & O \end{array}\right], \qquad (r = 1, 2, \ldots, k). \tag{51}$$

All that remains is the evaluation of the η's. From (50) and (51),

$$(52) \quad U^{-1}C^*CU = \operatorname{diag}[\xi_1^2, \xi_2^2, \ldots, \xi_p^2, \eta_1^2|\lambda_1|^2, \eta_1^{-2}|\lambda_1|^2, \ldots, \eta_k^2|\lambda_k|^2, \eta_k^{-2}|\lambda_k|^2].$$

Let the ξ's be ordered so that $\xi_1 \leqslant \xi_2 \leqslant, \ldots, \leqslant \xi_p$ and the λ's so that $|\lambda_1| \leqslant |\lambda_2| \leqslant, \ldots, \leqslant |\lambda_k|$. Assume the eigenvalues of C^*C to be ordered according to the scheme

$$\xi_1^2, \xi_2^2, \ldots, \xi_p^2, \alpha_{11}^2, \alpha_{12}^2, \alpha_{21}^2, \alpha_{22}^2, \ldots, \alpha_{k1}^2, \alpha_{k2}^2$$

in which all α's > 0 and

$$\alpha_{11}\alpha_{12} \leqslant \alpha_{21}\alpha_{22} \leqslant, \ldots, \leqslant \alpha_{k1}\alpha_{k2}.$$

Then,

$$(53) \qquad \eta_r = \sqrt{\left(\frac{\alpha_{r1}}{\alpha_{r2}}\right)}, \qquad (r = 1, 2, \ldots, k).$$

In other words the η's are uniquely determined by C, Q.E.D.

The reader shuold note that the above corollary does not necessarily include the cases $C = C'$ or $C = -C'$ which in general require separate arguments.

References

1. L. K. Hua, *On the theory of automorphic functions of a matrix variable I-geometrical basis*, Amer. J. Math., *66* (1944).
2. I. Schur, *Ein Satz ueber quadratische Formen mit komplexen Koeffizienten*, Amer. J. Math., *67* (1942), 472.
3. J. W. Stander and N. A. Wiegman, *Canonical forms for certain matrices under unitary congruence*, Can. J. Math., *12* (1960).
4. D. C. Youla, *Direct single frequency synthesis from a prescribed scattering matrix*, IRE Transactions of the Professional Group on Circuit Theory, vol. CT-6, no. 4 (December, 1959).
5. ——— *Some new and useful matrix results*, Mem. 34, Polytechnic Institute of Brooklyn, Microwave Research Institute (March 7, 1960).

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